FINITE ORDER VANISHING OF BOUNDARY VALUES OF HOLOMORPHIC MAPPINGS

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ABSTRACT. Suppose that f is a holomorphic function on a half-disc in the complex plane that extends continuously to the diameter E , such that the extension maps E to a double cone with the vertex at the origin. If the extension $f|_E$ has an isolated zero at the origin, then f vanishes to finite order at 0.

1. INTRODUCTION

Let $\Delta^+ = \{z \in \mathbb{C} : |z| < 1, \operatorname{Re} z > 0\}$ be a half-disc, $E = \{iy : -1 < y < 1\}$ its diameter, and $\Gamma = \{z \in \mathbb{C} : z = r e^{i\varphi}, \varphi \in [\varphi_1, \varphi_2], r \in \mathbb{R}\}$ a double cone in the complex plane with $\varphi_2 - \varphi_1 < \pi$. Let f be a function defined on Δ^+ . If $f(z) = o(z^n)$ for all integers $n \in \mathbb{Z}_+$, then we say that f vanishes to infinite order at 0. We shall prove the following result:

Theorem. *Suppose that f is a holomorphic function on Δ^+ that extends continuously to the diameter E , such that the extension maps E to Γ . If the extension $f|_E$ has an isolated zero at the origin, then f vanishes to finite order at 0.*

Results in this direction have been obtained recently by Bell and Lempert [2] and Alinhac, Baouendi, and Rothschild [1], where the role of a cone took a C^∞ smooth curve.

The idea of the proof is as follows. The path $f|_E$ lies in a cone, so it cannot wind around the origin as is the case for $f(z) = e^{-1/\sqrt{z}}$ which extends to a C^∞ function on $\overline{\Delta^+}$ and vanishes to infinite order at 0. We shall develop the notion of index of a continuous curve that is not necessarily closed. The theorem will follow from Lemmas 1 and 2 below.

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2. INDEX

Let $\gamma : [\alpha, \beta] \rightarrow \mathbb{C} \setminus \{0\}$ be a piecewise continuously differentiable curve in the plane. If γ is closed then the index of γ with respect to 0 is the integer

$$w(\gamma) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z} = \frac{1}{2\pi i} \int_{\alpha}^{\beta} \frac{\gamma'(t)}{\gamma(t)} dt.$$

If the curve γ is not closed, then the number $w(\gamma)$ defined above is real if and only if the endpoints of γ are at the same distance from the origin.

We define *the index of γ with respect to 0* to be the real number

$$\text{Ind}(\gamma) = \Re w(\gamma).$$

The number $2\pi \text{Ind}(\gamma)$ is the net increase of the argument of $\gamma(t)$. If two paths in $\mathbb{C} \setminus \{0\}$ have the same endpoints, then the difference of their indexes is an integer.

This index has the following properties:

$$(2.1) \quad \text{Ind}(\gamma) = \text{Ind}(\gamma/|\gamma|).$$

$$(2.2) \quad \text{Ind}(\gamma) = 0, \text{ if } \gamma \text{ lies on a ray from the origin.}$$

$$(2.3) \quad |\text{Ind}(\gamma)| < 1/2, \text{ if } \gamma \text{ lies in a half-plane } \{z : \Re(az) > 0\} \text{ through the origin.}$$

If $\gamma : [\alpha, \beta] \rightarrow \mathbb{C} \setminus \{0\}$ is only a continuous path, then let $\delta > 0$ be its distance from the origin and let $\tilde{\gamma} : [\alpha, \beta] \rightarrow \mathbb{C} \setminus \{0\}$ be any continuously differentiable path satisfying

$$(i) \quad |\tilde{\gamma}(t) - \gamma(t)| < \delta \text{ for all } t \in [\alpha, \beta], \text{ and}$$

$$(ii) \quad \tilde{\gamma}(\alpha) = \gamma(\alpha), \quad \tilde{\gamma}(\beta) = \gamma(\beta).$$

We define *the index of γ with respect to 0* to be the number

$$\text{Ind}(\gamma) = \text{Ind}(\tilde{\gamma}).$$

A homotopy argument shows that this number is independent of the choice of $\tilde{\gamma}$ provided that (i) and (ii) hold. Properties (2.1), (2.2), and (2.3) remain valid.

3. Lemma 1

Lemma 1. *Let $f : \Delta^+ \rightarrow \mathbb{C}$ be as in the theorem. If the function f has an isolated zero at the origin in $\overline{\Delta}^+$ then f cannot vanish to infinite order at 0.*

Proof. Suppose the lemma is not true. Shrinking Δ^+ we may assume that the origin is the only zero of f on $\overline{\Delta}^+$. Let $\gamma_r(t) = r e^{it}$, $t \in [-\pi/2, \pi/2]$, $r \in (0, 1]$, and let $I(r) = \text{Ind}(f \circ \gamma_r)$ be the index of the path $f \circ \gamma_r$ with respect to the origin. We shall prove

$$(3.1) \quad \lim_{r \searrow 0} \frac{1}{\ln 1/r} \int_r^1 \frac{I(r)}{r} dr = \infty.$$

Suppose that (3.1) is valid. Let $\gamma \subset \overline{\Delta}^+$ be the closed path $\gamma = \gamma_r + l_r^+ + (-\gamma_1) + l_r^-$, where l_r^+ , l_r^- are straight connections. See Figure 1.

If we shrink γ to $\tilde{\gamma}$ that lies in Δ^+ , we may apply Cauchy's theorem to f'/f and get

$$(3.2) \quad 0 = \Re \frac{1}{2\pi i} \int_{\tilde{\gamma}} \frac{f'}{f} dz = I(r) + \text{Ind}(f \circ l_r^+) - I(1) + \text{Ind}(f \circ l_r^-).$$

FIGURE 1

The paths $f \circ l_r^\pm$ lie in a cone Γ , so their indexes are bounded by (2.3). From (3.2) we conclude that there is a constant $C < \infty$ such that $|I(r)| \leq C$ for all $r \in (0, 1]$. Hence the averages in (3.1) are bounded and we get a contradiction. To prove (3.1) we map Δ^+ conformally onto the strip $S = \{w \in \mathbb{C} : \Re w < 0, |\Im w| < \pi/2\}$ by the mapping $w = \ln z$. Since Δ^+ is simply connected, there is a function Φ , holomorphic on Δ^+ and continuous on $\bar{\Delta}^+ \setminus \{0\}$, such that

$$f(z) = e^{\Phi(z)}, \quad z \in \bar{\Delta}^+ \setminus \{0\}.$$

For all $\alpha \in (0, \pi/2)$ we have for the path $\gamma_r^\alpha(t) = r e^{it}$, $t \in [-\alpha, \alpha]$,

$$\begin{aligned} \text{Ind}(f \circ \gamma_r^\alpha) &= \Re \frac{1}{2\pi i} \int_{\gamma_r^\alpha} \frac{f'}{f} dz = \Re \frac{1}{2\pi i} \int_{\gamma_r^\alpha} \Phi' dz \\ &= \Re \frac{1}{2\pi i} [\Phi(r e^{i\alpha}) - \Phi(r e^{-i\alpha})]. \end{aligned}$$

In the limit as $\alpha \nearrow \pi/2$ we get

$$I(r) = \Re \frac{1}{2\pi i} [\Phi(r e^{i\pi/2}) - \Phi(r e^{-i\pi/2})].$$

Define $\Psi(w) = \Phi(e^w) = u(x, y) + iv(x, y)$, where $w = x + iy$ is a point in $\bar{S}$. With this notation we have

$$I(r) = \frac{1}{2\pi} [v(\ln r, \pi/2) - v(\ln r, -\pi/2)] = \frac{1}{2\pi} J_{\pi/2}(\ln r)$$

where we have defined

$$(3.3) \quad J_\alpha(x) = v(x, \alpha) - v(x, -\alpha), \quad x \leq 0, \quad 0 < \alpha \leq \pi/2.$$

Using the Cauchy-Riemann equations we obtain

$$J_\alpha(x) = \int_{-\alpha}^{\alpha} \frac{\partial v}{\partial y}(x, y) dy = \int_{-\alpha}^{\alpha} \frac{\partial u}{\partial x}(x, y) dy$$

and for $c < 0$

$$\begin{aligned} \int_c^0 J_\alpha(x) dx &= \int_{-\alpha}^\alpha dy \int_c^0 \frac{\partial u}{\partial x} dx = \int_{-\alpha}^\alpha u(0, y) dy - \int_{-\alpha}^\alpha u(c, y) dy \\ &\geq K - \int_{-\alpha}^\alpha u(c, y) dy, \end{aligned}$$

where K is a constant independent of α and c . In the limit as $\alpha \nearrow \pi/2$ we get

$$(3.4) \quad \int_c^0 J_{\pi/2}(x) dx \geq K - \int_{-\pi/2}^{\pi/2} u(c, y) dy, \quad c < 0.$$

The infinite order vanishing condition $f(z) = o(z^k)$ for all $k \in \mathbb{N}$ implies

$$\begin{aligned} 0 &= \lim_{\substack{z \rightarrow 0 \\ z \in \bar{\Delta}^+}} \left| \frac{f(z)}{z^k} \right| = \lim_{\substack{z \rightarrow 0 \\ z \in \bar{\Delta}^+}} \left| \frac{e^{\Phi(z)}}{e^{k \ln z}} \right| = \lim_{\substack{z \rightarrow 0 \\ z \in \bar{\Delta}^+}} |e^{\Psi(\ln z) - k \ln z}| \\ &= \lim_{\substack{z \rightarrow 0 \\ z \in \bar{\Delta}^+}} e^{\Re[\Psi(\ln z) - k \ln z]} = \lim_{\substack{w \rightarrow \infty \\ w \in \bar{S}}} e^{u(x, y) - kx}. \end{aligned}$$

Hence for every $k \in \mathbb{N}$ there is a constant $x_0 < 0$ such that $u(x, y) \leq kx$ for all $x \leq x_0$, $y \in [-\pi/2, \pi/2]$. This together with (3.4) gives

$$\int_c^0 J_{\pi/2}(x) dx \geq K - \pi kc,$$

so

$$\liminf_{c \rightarrow -\infty} \frac{1}{|c|} \int_c^0 J_{\pi/2}(x) dx \geq \pi k \quad \text{for every } k \in \mathbb{N}.$$

For $c = \ln r$ we get

$$\frac{1}{|c|} \int_c^0 J_{\pi/2}(x) dx = \frac{1}{\ln 1/r} \int_r^1 J_{\pi/2}(\ln r) \frac{dr}{r} = \frac{2\pi}{\ln 1/r} \int_r^1 \frac{I(r)}{r} dr$$

and (3.1) is proved.

4. LEMMA 2

Lemma 2. Let $f: \Delta^+ \rightarrow \mathbb{C}$ be as in the theorem. Then the origin $0 \in \mathbb{C}$ is not the limit point of zeros of f in Δ^+ .

Proof. We may suppose that $f|_E$ vanishes only at the origin and

$$(4.1) \quad |f| \leq 1 \quad \text{on } \bar{\Delta}^+.$$

Suppose that the lemma does not hold. Let (r_n) be a decreasing sequence of positive numbers, such that all zeros of f lie on half-circles in Δ^+ with centers at origin. Each of these half-circles contains only finitely many, say $\kappa(n)$, zeros of f , counted with multiplicities. Let

$$A_n = \{r e^{i\varphi} : \varphi \in [-\pi/2, \pi/2], r_{n+1} < r < r_n\}$$

be the semi-annulus bounded with half-circles of radius r_n and r_{n+1} . As in the previous proof we take the path $\gamma_r(t) = re^{it}$, $t \in [-\pi/2, \pi/2]$, $r \in (0, 1]$. We shall assume that $r \neq r_n$, so the path does not contain any zeros of f . Let $I(r) = \text{Ind}(f \circ \gamma_r)$ be the index of $f \circ \gamma_r$ with respect to the origin.

First we can show the existence of one sided limits of the index $I(r)$ at points r_n . Fix $n \in \mathbb{N}$ and write f in the form

$$f(z) = (z - a_1)^{l_1} \cdots (z - a_j)^{l_j} \psi(z)$$

where $a_1, \dots, a_j$ are all the zeros of f on the half-circle of radius r_n ($\sum_1^j l_k = \kappa(n)$) and ψ is a continuous function without zeros on closure of $A_{n-1} \cup A_n$, holomorphic in the interior. For all points z in $A_{n-1} \cup A_n$ we have

$$\frac{f'(z)}{f(z)} = \sum_{k=1}^j \frac{l_k}{z - a_k} + \frac{\psi'(z)}{\psi(z)}.$$

It follows that

$$(4.2) \quad I(r) = \sum_{k=1}^j l_k \text{Ind}(\gamma_r - a_k) + \text{Ind}(\psi \circ \gamma_r).$$

The last term is a continuous function of r on the interval (r_{n+1}, r_n) . At the point $r = |a|$ the index $\text{Ind}(\gamma_r - a)$ has left and right limit equal to $-1/4$, $3/4$ respectively. From (4.2) we conclude that

$$\begin{aligned} I(r_n-) &= \lim_{r \nearrow r_n} I(r) = -\kappa(n)/4 + \text{Ind}(\psi \circ \gamma_{r_n}), \\ I(r_n+) &= \lim_{r \searrow r_n} I(r) = 3\kappa(n)/4 + \text{Ind}(\psi \circ \gamma_{r_n}), \end{aligned}$$

and

$$(4.3) \quad I(r_n+) = I(r_n-) + \kappa(n), \quad n \in \mathbb{N}.$$

In the semi-annulus A_n we construct a closed path as in the proof of Lemma 1 and get

$$(4.4) \quad I(r_n-) - I(r_{n+1}+) + \text{Ind}(f \circ l_n^+) + \text{Ind}(f \circ l_n^-) = 0.$$

Since the straight line connections $l_n^\pm$ are mapped by f into the cone Γ , we obtain

$$(4.5) \quad |I(r'') - I(r')| < 1, \quad r_{n+1} < r' < r'' < r_n.$$

From (4.3) and (4.4) we conclude

$$(4.6) \quad \kappa(n) = I(r_n+) - I(r_{n+1}+) + s(n), \quad n \in \mathbb{N},$$

where $s(n) = \text{Ind}(f \circ l_n^+) + \text{Ind}(f \circ l_n^-)$. Summing the first m relations (4.6) we get

$$I(r_1+) - I(r_{m+1}+) + \sum_{n=1}^m s(n) = \sum_{n=1}^m \kappa(n).$$

Since $\sum s(n)$ is the sum of indexes of two paths lying in the cone Γ , it is bounded by one, so

$$\lim_{n \rightarrow \infty} I(r_n+) = -\infty.$$

From this and (4.5) we conclude

$$(4.7) \quad \lim_{\substack{r \searrow 0 \\ r \notin \{r_n\}}} I(r) = -\infty.$$

Hence there is a constant $r_0 \in (0, 1)$ such that

$$(4.8) \quad I(r) \leq -1 \quad \text{for all } r \in (0, r_0] \setminus \bigcup_1^\infty \{r_n\}.$$

We shall assume that f is a continuously differentiable function on $\bar{\Delta}^+$. In the continuous setting the proof has to be modified as in Lemma 2; we introduce $J_\alpha(x)$ by (3.3) and take the limit as $\alpha \nearrow \pi/2$.

As in the previous proof we map Δ^+ onto a strip S by logarithm $w = \ln z$ and get a continuously differentiable function Φ on $\bigcup A_n$, holomorphic in the interior, such that $f(z) = e^{\Phi(z)}$. We also have

$$I(r) = \Re e \frac{1}{2\pi i} [\Phi(r e^{i\pi/2}) - \Phi(r e^{-i\pi/2})], \quad r \in (0, 1) \setminus \bigcup \{r_n\},$$

and

$$(4.9) \quad I(r) = \frac{1}{2\pi} [v(\ln r, \pi/2) - v(\ln r, -\pi/2)] = \frac{1}{2\pi} J(\ln r)$$

where $\Psi(w) = \Phi(e^w) = u(x, y) + iv(x, y)$ and $J(x) = v(x, \pi/2) - v(x, -\pi/2) = \int_{-\pi/2}^{\pi/2} (\partial u / \partial x) dy$. From (4.9) we see that function J is piecewise continuous and has one sided limits at points $\ln r_n$. The integral $\int_c^{\ln r_0} J(x) dx$ exists for every $c < \ln r_0$. On the interval $[c, \ln r_0]$ the integrand $J(x)$ has only finitely many discontinuities. In the following calculation we assume that there is only one such point β :

$$(4.10) \quad \begin{aligned} \int_c^{\ln r_0} J(x) dx &= \lim_{\varepsilon \searrow 0} \left[\int_c^{\beta-\varepsilon} J(x) dx + \int_{\beta+\varepsilon}^{\ln r_0} J(x) dx \right] \\ &= \int_{-\pi/2}^{\pi/2} u(\ln r_0, y) dy - \int_{-\pi/2}^{\pi/2} u(c, y) dy \\ &\quad + \lim_{\varepsilon \searrow 0} \int_{-\pi/2}^{\pi/2} [u(\beta - \varepsilon, y) - u(\beta + \varepsilon, y)] dy. \end{aligned}$$

We can show that the last limit equals zero. We first observe

$$(4.11) \quad e^{u(x, y)} = |\exp(\Phi(e^w))| = |f(e^w)|$$

and

$$(4.12) \quad u(\beta - \varepsilon, y) - u(\beta + \varepsilon, y) = \ln \left| \frac{f(e^{\beta - \varepsilon + iy})}{f(e^{\beta + \varepsilon + iy})} \right|.$$

Then we factorize all zeros of f on the half-circle of radius e^β :

$$f(z) = \prod_{j=1}^k (z - e^{\beta + i\alpha_j}) \varphi(z)$$

where $\varphi(e^w)$ is without zeros on the rectangle $[c, \ln r_0] \times [-\pi/2, \pi/2]$. The equation (4.12) is now

$$u(\beta - \varepsilon, y) - u(\beta + \varepsilon, y) = -k\varepsilon + \ln \left| \frac{\varphi(e^{\beta - \varepsilon + iy})}{\varphi(e^{\beta + \varepsilon + iy})} \right|$$

since

$$\left| \frac{e^{\beta - \varepsilon + iy} - e^{\beta + i\alpha_j}}{e^{\beta + \varepsilon + iy} - e^{\beta + i\alpha_j}} \right| = e^{-\varepsilon}$$

and the last limit in (4.10) is equal to zero. Finally we use (4.10), (4.11), and (4.1)

$$\begin{aligned} \int_c^{\ln r_0} J(x) dx &= \int_{-\pi/2}^{\pi/2} u(\ln r_0, y) dy - \int_{-\pi/2}^{\pi/2} u(c, y) dy \\ &\geq \int_{-\pi/2}^{\pi/2} u(\ln r_0, y) dy = K. \end{aligned}$$

However, if we use (4.8) and (4.9), this gives a contradiction:

$$K \leq \int_{\ln r}^{\ln r_0} J(x) dx \leq -2\pi \ln \frac{r_0}{r} \xrightarrow{r \searrow 0} -\infty.$$

Lemma 2 is proved.

5. REMARKS

It is interesting that our index $I(r)$ of the path $f \circ \gamma_r$ in Lemmas 1 and 2 behaves in entirely different ways, as we see in (3.1) and (4.7). In Lemma 1 we could show only the divergence of averages in (3.1), not of the index $I(r)$ itself.

Let f be of smooth to infinite order on the closure $\bar{\Delta}^+$ and holomorphic in the interior Δ^+ . If f has a sequence of zeros $(z_n) \subset \Delta^+$ converging to 0, then f must have a zero of infinite order at 0.

We do not know whether the theorem is true if the origin is not an isolated zero of $f|_E$ on the diameter E of the half-disc Δ^+ .

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