ALGEBRAS OF FINITE SELF-INJECTIVE DIMENSION

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ABSTRACT. Let A be an artin algebra. Then A has finite self-injective dimensions on both sides if and only if every finitely generated left A -module has finite Gorenstein dimension.

Two decades ago, Auslander and Bridger [2] showed that a commutative noetherian local ring A is a Gorenstein ring if and only if every finitely generated A -module has finite Gorenstein dimension. In this paper, we will develop their arguments and apply obtained results to artin algebras. We will prove the following:

Theorem. *Let A be an artin algebra. Then $\text{inj dim}_A A = \text{inj dim } A_A < \infty$ if and only if every finitely generated left A -module has finite Gorenstein dimension.*

In the following, we will work over a left and right noetherian ring A . We denote by $()^*$ both A -dual functors. Right A -modules are considered as left A^{op} -modules, where A^{op} stands for the opposite ring of A , and notations for left A -modules are used also for left A^{op} -modules. Denote by $\text{mod } A$ the category of all finitely generated left A -modules. For $n > 0$, a module $M \in \text{mod } A$ is said to be an n th syzygy if there is an exact sequence $0 \rightarrow M \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0$ in $\text{mod } A$ with the P_i projective. Let $\text{syz}^n A$ be the class of all $M \in \text{mod } A$ that are n th syzygies, and, for the sake of convenience, put $\text{syz}^0 A = \text{mod } A$ (i.e., every $M \in \text{mod } A$ is considered as a 0th syzygy of itself). Also, a module $M \in \text{mod } A$ is said to have reduced grade at least n , written $\text{reduced grade } M \geq n$, if $\text{Ext}_A^i(M, A) = 0$ for $1 \leq i < n$. Note that $\text{reduced grade } M \geq 1$ for all $M \in \text{mod } A$. Finally, a module $M \in \text{mod } A$ is said to have Gorenstein dimension zero, written $\text{G-dim } M = 0$, if it is reflexive and $\text{reduced grade } M = \text{reduced grade } M^* = \infty$. Then, for $n > 0$, M is said to have Gorenstein dimension at most n , written $\text{G-dim } M \leq n$, if there is an exact sequence $0 \rightarrow M_n \rightarrow \cdots \rightarrow M_0 \rightarrow M \rightarrow 0$ in $\text{mod } A$ with $\text{G-dim } M_i = 0$ for $0 \leq i \leq n$. Note that $\text{G-dim } M < \infty$ implies $\text{Ext}_A^i(M, A) = 0$ for $i > \text{G-dim } M$ (see Auslander and Bridger [2] for details).

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PROOF OF THEOREM

We divide the proof into several steps.

Lemma 1. *Let $M \in \text{mod } A$ have infinite reduced grade. Then $\text{G-dim } M < \infty$ implies $\text{G-dim } M = 0$.*

Proof. See Auslander and Bridger [2, p. 95].

Lemma 2. *Let $P_1 \xrightarrow{f} P_0 \rightarrow M \rightarrow 0$ be an exact sequence in $\text{mod } A$ with the P_i projective. Put $N = \text{Cok } f^*$. Then M is reflexive if and only if reduced grade $N \geq 3$.*

Proof. See Auslander [1, Proposition 6.3].

Lemma 3. *Let $M \in \text{mod } A$ have reduced grade $\geq n$. Let $P_n \xrightarrow{f} P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0$ be an exact sequence in $\text{mod } A$ with the P_i projective. Then $N = \text{Cok } f^*$ has reduced grade $\geq n$.*

Proof. We have an exact sequence $P_0^* \rightarrow \cdots \rightarrow P_{n-1}^* \rightarrow P_n^* \rightarrow N \rightarrow 0$ in $\text{mod } A^{\text{op}}$ with the P_i^* projective. Since each P_i is reflexive, if $(\)^*$ is applied, the induced sequence $0 \rightarrow N^* \rightarrow P_n^{**} \rightarrow P_{n-1}^{**} \rightarrow \cdots \rightarrow P_0^{**}$ is exact, which implies reduced grade $N \geq n$.

Lemma 4. *For any $n \geq 1$, the following are equivalent:*

- (1) *Every $M \in \text{mod } A$ of reduced grade $\geq n$ is reflexive.*
- (2) *Every $N \in \text{mod } A^{\text{op}}$ of reduced grade $\geq n$ has infinite reduced grade.*

Proof. (1) $\Rightarrow$ (2). Let $N \in \text{mod } A^{\text{op}}$ have reduced grade $\geq n$. Let $P_n \xrightarrow{f} P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow N \rightarrow 0$ be an exact sequence in $\text{mod } A^{\text{op}}$ with the P_i projective. Put $L = \text{Cok } f$ and $M = \text{Cok } f^*$. Then by Lemma 3 reduced grade $M \geq n$ and M is reflexive. Thus by Lemma 2 $\text{Ext}_A^n(N, A) \simeq \text{Ext}_A^1(L, A) = 0$ and we get reduced grade $N \geq n + 1$. Now, by applying this argument to syzygies of N , we conclude that N has infinite reduced grade.

(2) $\Rightarrow$ (1). Let $M \in \text{mod } A$ have reduced grade $\geq n$. Let $P_n \xrightarrow{f_n} P_{n-1} \rightarrow \cdots \xrightarrow{f_1} P_0 \rightarrow M \rightarrow 0$ be an exact sequence in $\text{mod } A$ with the P_i projective. Put $L = \text{Cok } f_1^*$ and $N = \text{Cok } f_n^*$. Note that L is an $(n - 1)$ th syzygy of N . Since by Lemma 3 reduced grade $N \geq n$, we have reduced grade $N = \infty$ and by Lemma 2, M is reflexive.

Lemma 5. *For any $n \geq 0$, the following are equivalent:*

- (1) *$\text{inj dim}_A A \leq n$ and every $M \in \text{mod } A$ of infinite reduced grade is reflexive.*
- (2) *$\text{G-dim } M = 0$ for all $M \in \text{syz}^n A$.*

Proof. (1) $\Rightarrow$ (2). Let $M \in \text{syz}^n A$. Then $\text{reduced grade } M = \infty$ and M is reflexive. Let $P_{n+1} \xrightarrow{f} P_n \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0$ be an exact sequence in $\text{mod } A$ with the P_i projective. Put $N = \text{Cok } f^*$. Note that M^* is an $(n+2)$ th syzygy of N . Since by Lemmas 3 and 4 $\text{reduced grade } N = \infty$, we get $\text{reduced grade } M^* = \infty$. Thus $\text{G-dim } M = 0$.

(2) $\Rightarrow$ (1). The first assertion is obvious. Since $\text{G-dim } M \leq n$ for all $M \in \text{mod } A$, the last assertion follows from Lemma 1.

Lemma 6. *Suppose $\text{inj dim } A_A < \infty$. Then $\text{G-dim } M = 0$ for all $M \in \text{mod } A$ of infinite reduced grade.*

Proof. Let $M \in \text{mod } A$ have infinite reduced grade. Since $M^* \in \text{syz}^n A^{\text{op}}$ for all $n \geq 0$, $\text{inj dim } A_A < \infty$ implies $\text{reduced grade } M^* = \infty$. Finally, by Lemma 4, $\text{inj dim } A_A < \infty$ implies that M is reflexive.

Lemma 7. *Suppose $\text{inj dim } A_A \leq n$. Then $\text{inj dim } M \leq n$ for all $M \in \text{mod } A$ of finite projective dimension.*

Proof. Note that for an exact sequence $0 \rightarrow M_3 \rightarrow M_2 \rightarrow M_1 \rightarrow 0$ we have $\text{inj dim } M_1 \leq \max\{\text{inj dim } M_2, \text{inj dim } M_3\}$. By induction on $\text{proj dim } M < \infty$, the assertion follows.

Lemma 8. *Let $M \in \text{mod } A$ be reflexive with $\text{reduced grade } M^* = \infty$. Then $\text{inj dim } M < \infty$ implies that M is projective.*

Proof. We have an exact sequence $0 \rightarrow M \xrightarrow{f} P_0 \rightarrow P_1 \rightarrow \cdots$ in $\text{mod } A$ with the P_i projective. Since $\text{inj dim } M < \infty$ implies f splitting, the assertion follows.

Lemma 9. *Let $n \geq 0$. Suppose $\text{G-dim } L = 0$ for all $L \in \text{syz}^n A$. Then, for any $M \in \text{mod } A$, the following are equivalent:*

- (1) $\text{proj dim } M < \infty$.
- (2) $\text{proj dim } M \leq n$.
- (3) $\text{inj dim } M < \infty$.
- (4) $\text{inj dim } M \leq n$.

Proof. By Lemma 5 we have $\text{inj dim } A_A \leq n$. Thus by Lemma 7 (1) $\Rightarrow$ (4). It only remains to prove (3) $\Rightarrow$ (2). Suppose $\text{inj dim } M < \infty$. Let L be an n th syzygy of M . Then $\text{inj dim } L < \infty$. Since $\text{G-dim } L = 0$, by Lemma 8 L is projective; namely, $\text{proj dim } M \leq n$.

Lemma 10. *Suppose there is some $E \in \text{mod } A$ which is an injective cogenerator for left A -modules. Suppose $\text{inj dim } A_A \leq n$. Then the following are equivalent:*

- (1) $\text{inj dim } A_A \leq n$.
- (2) Every $M \in \text{mod } A$ of infinite reduced grade is reflexive.
- (3) $\text{inj dim } M \leq n$ for all $M \in \text{mod } A$ of finite injective dimension.

Proof. By Lemma 4, $(1) \Rightarrow (2)$, and by Lemmas 5 and 9, $(2) \Rightarrow (3)$.

$(3) \Rightarrow (1)$. Let $\cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \rightarrow E \rightarrow 0$ be an exact sequence in $\text{mod } A$ with the P_i projective. Put $M_i = \text{Im } f_i$ for $i \geq 1$. Note that $\text{inj dim } M_i < \infty$ thus $\text{inj dim } M_i \leq n$ for all $i \geq 1$. Hence $\text{Ext}_A^{n+1}(E, M_{n+1}) = 0$, which implies $\text{proj dim } E \leq n$. The assertion now follows from Cartan and Eilenberg [3, Chapter VI, Proposition 5.3].

Proof of theorem. "If" part. There is some $n \geq 0$ such that $\text{G-dim } M \leq n$ for all simple modules $M \in \text{mod } A$. Then $\text{inj dim}_A A \leq n$. Thus, by Lemmas 1 and 10, $\text{inj dim } A_A \leq n$ and by Zaks [4, Lemma A], $\text{inj dim}_A A = \text{inj dim } A_A \leq n$.

"Only if" part. Suppose $\text{inj dim}_A A = \text{inj dim } A_A \leq n$. Then by Lemma 6, $\text{G-dim } M = 0$ for all $M \in \text{syzy}^n A$.

REFERENCES

1. M. Auslander, *Coherent functors*, Proc. Conf. Cat. Algebra, Springer, Berlin, 1966, pp. 189–231.
2. M. Auslander and M. Bridger, *Stable module theory*, Mem. Amer Math. Soc., no. 94, Amer. Math. Soc., Providence, RI, 1969.
3. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, 1956.
4. A. Zaks, *Injective dimension of semiprimary rings*, J. Algebra **13** (1969), 73–86.

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