ω_1 CAN BE A NONTRIVIAL AUTOMORPHISM GROUP

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ABSTRACT. Under the axiom “there are ω_1 cofinal Cohen reals,” the symmetric group on ω is the nontrivial automorphism group of a thin-tall superatomic Boolean algebra. Certain product groups are also, under the same axiom, nontrivial automorphism groups of thin-tall superatomic Boolean algebras.

1. INTRODUCTION

A superatomic Boolean algebra is one in which every quotient algebra is atomic. Such an algebra is a continuous increasing union of ideals $\{J_\alpha : \alpha \leq \beta\}$, where $J_{\alpha+1}$ is generated by J_α together with a set of representatives of the atoms of B/J_α . β is always a successor ordinal, called the height of B ; $\sup_{\alpha < \beta} |J_{\alpha+1}/J_\alpha|$ is called the width of B . B is thin-tall iff its width is ω and its height is $\omega_1 + 1$. An automorphism f is trivial iff some f_α is the identity on some B/J_α , where $\alpha + 1 < \text{height}(B)$ and $f_\alpha(b) = b/J_\alpha$. The trivial automorphisms form a normal subgroup $N(B)$ of the automorphism group $\text{Aut}(B)$. The nontrivial automorphism group $G(B) = \text{Aut}(B)/N(B)$.

Dow and Simon [DS] proved that every countable group is $G(B)$ for some thin-tall superatomic Boolean algebra. In [Ro2] it is shown that a class of groups, including both the free group and the free Abelian group on ω_1 generators, can consistently be $G(B)$ for some thin-tall superatomic Boolean algebra. The axiom used was “there are ω_1 cofinal Cohen reals.” This is the same axiom used in this paper, and it is defined forthwith.

Definition 1. There are ω_1 cofinal Cohen reals iff there are sequences $\{M_\alpha : \alpha < \omega_1\}$ and $\{G_\alpha : \alpha < \omega_1\}$, where

- (1) for each α , G_α is $\mathbf{P}$ -generic over M_α , where $\mathbf{P}$ is the usual partial order adding one Cohen real,
- (2) M_α is an elementary submodel of $HC =$ the collection of hereditarily countable sets, and
- (3) $HC = \bigcup_{\alpha < \omega_1} M_\alpha$.

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In particular, CH implies that there are ω_1 cofinal Cohen reals, and MA refutes it.

Although the abstract is stated in the Boolean algebraic form, this particular proof is easier to give in its topological form.

Thus, by Stone duality we are interested in locally compact scattered Hausdorff spaces X , where X is the continuous union of $\{X_\alpha : \alpha < \omega_1\}$, each $X_{\alpha+1} - X_\alpha$ is the set of isolated points of $X - X_\alpha$, and each $|X_{\alpha+1} - X_\alpha| = \omega$. Such a space will be called thin-tall. An autohomeomorphism f of X must satisfy $f[X_{\alpha+1} - X_\alpha] = X_{\alpha+1} - X_\alpha$ for all α . A trivial autohomeomorphism in this context is one which moves only countably many points of X . If $N(X)$ is the group of trivial autohomeomorphisms of X and $\text{Aut}(X)$ is the group of autohomeomorphisms of X , then $G(X) = \text{Aut}(X)/N(X)$. X is semirigid iff every nontrivial autohomeomorphism is the identity of a countable set.

Theorem 2. *Suppose there are ω_1 cofinal Cohen reals. Then $\omega!$ is the nontrivial autohomeomorphism group $G(X)$ of a thin-tall locally compact Hausdorff space. Equivalently, $\omega!$ is $G(B)$ for a thin-tall superatomic Boolean algebra B .*

For the rest of this paper we assume the following axiom: there are ω_1 cofinal Cohen reals.

2. SKETCH OF PROOF AND NOTATION

First we will construct a first countable thin-tall locally compact semirigid Hausdorff space $X = A \times \omega_1$, where $A = \{c \subset \omega : 1 \leq |c| \leq 2\}$.

Some notation:

- (i) For $\alpha \in \omega_1$, $X|\alpha$ denotes $A \times \alpha$.
- (ii) For $c \in A$, X_c denotes $\{c\} \times \omega_1$.

X will be constructed so each nonempty $X|\alpha$ is open dense. The final space S is the disjoint sum of countably many copies of X .

More notation:

- (iii) The n th copy of X in S is denoted by $X(n)$.
- (iv) If $x = (c, \alpha)$ is a point in X , then $x(n) = (c, \alpha, n)$ is the copy of x in $X(n)$.
- (v) If Y is a subset of X , then $Y(n)$ is the copy of Y in $X(n)$.
- (vi) $X_c(n)$, the copy of X_c in $X(n)$, is also denoted by $X_{c,n}$.

The symbol $\equiv$ denotes equality modulo the ideal of countable sets.

X will be constructed to satisfy the following properties:

Property A. Let f be an autohomeomorphism of S . Then for every c, n there is some r so that $f[X_{c,n}] \equiv X_{c,r}$.

Remark. Property A implies that X is semirigid.

Property B. Let f be an autohomeomorphism of S . If $f[X_{c,n}] \equiv X_{c,r}$ and $f[X_{d,n}] \equiv X_{d,s}$, then $r = s$.

Suppose these properties are satisfied. Then for any autohomeomorphism f of S and any n , $f[X(n)] \equiv X(m)$ for some m and, by the remark, for all $c \in A$, $f(c, \delta, n) = (c, \delta, m)$ for all but countably many δ . Define $\varphi_f(n) = m$ iff $f[X(n)] \equiv X(m)$. Note that the function taking f to φ_f is a homomorphism from $\text{Aut}(X)$ onto $\omega!$ whose kernel is $N(X)$. So $G(X)$ is isomorphic to $\omega!$.

3. CONSTRUCTING X

The topology on X will be constructed by inductively constructing the topologies on each $X|\alpha$, $\alpha < \omega_1$, where each $X|\alpha$ will be Hausdorff, first countable, and locally compact.

Fix α . Suppose we know the topology on $X|\alpha$. Let ψ_α be a 1-1 function from α into ω , i.e., ψ_α witnesses that α is countable. Since ψ_α , $X|\alpha$, and a basis $\mathcal{B}_\alpha$ for the topology on $X|\alpha$ are hereditarily countable, there is $M = M_{\beta_\alpha}$ with ψ_α , $X|\alpha$, and $\mathcal{B}_\alpha \in M$. Let G_{β_α} be a generic filter adding a Cohen real to M , as in condition (1) of Definition 1. We use G_{β_α} to construct a generic partition of $X|\alpha$ into compact clopen sets.

Remark. In this construction we use the fact that iterating Cohen forcing twice is equivalent to using it once: thus there are G , a filter adding a Cohen real to M , and H , a filter adding a Cohen real to $M[G]$ with $M[G_{\beta_\alpha}] = M[G][H]$.

The partial order $\mathbf{P} = \mathbf{P}_\alpha$ consists of all σ , where σ is a function, $\text{dom } \sigma = n$ for some n , each $\sigma(i)$ is a finite collection of compact clopen subsets of $X|\alpha$, $\sigma(i) \cap \sigma(j) = \emptyset$ for $i \neq j$, and $\bigcup \text{range } \sigma$ is pairwise disjoint. The order is $\sigma \leq \tau$ iff $\text{dom}(\sigma) \supset \text{dom}(\tau)$ and each $\sigma(i) \supset \tau(i)$.

$\mathbf{P}$ is countable, so forcing with $\mathbf{P}$ is equivalent to adding a Cohen real. The set $D_x = \{\sigma : x \in u \in \sigma(i) \text{ for some } u, i\}$ is dense. So if G is a generic filter on $\mathbf{P}$, and $P_i = P_{i,\alpha} = \bigcup \{\sigma(i) : \sigma \in G\}$, then the D_x 's force $\bigcup \{P_i : i < \omega\}$ to be a partition of $X|\alpha$, where if $i \neq j$, then $P_i \cap P_j = \emptyset$.

Now let G be a $\mathbf{P}$ -generic over M and let $\mathbf{Q} = \mathbf{Q}_\alpha$ be the countable partial order over $M[G]$ which adds, for each $c = \{i, j\} \in A$ (where i, j need not be distinct), a set $Q_{c,\alpha} = Q_c \subset P_i \cup P_j$, where the Q_c 's are pairwise disjoint. Thus a condition in $\mathbf{Q}$ is some σ , where $\text{dom}(\sigma)$ is a finite subset of A , $\sigma(\{i, j\})$ is a finite subset of $P_i \cup P_j$, and if $c \neq d$ then $\sigma(c)$ is disjoint from $\sigma(d)$.

By the remark, G_{β_α} adds $\{P_i : i < \omega\}$ and $\{Q_c : c \in A\}$ as above.

A basic neighborhood of the point (c, α) in $X|\alpha + 1$ consists of (c, α) together with the union of all but finitely many elements of Q_c . Points in $X|\alpha$ retain their old neighborhood basis. Because $\{P_i : i < \omega\}$ partitioned $X|\alpha$ into compact open sets, this extension to $X|\alpha + 1$ continues to be Hausdorff, first countable, and locally compact. So the final topology on X is Hausdorff, first countable, and locally compact.

For notational purposes, define $B_c = B_{c,\alpha} = \bigcup Q_{c,\alpha}$.

Remark. A set which intersects each B_c in at most one point does not converge in X .

Remark. If α is a limit, $\{\delta_n : n < \omega\}$ is an increasing sequence converging to α , and for each $n < \omega$ there is d_n , where $x_n = (d_n, \delta_n) \in B_c$, then $\{x_n : n < \omega\}$ converges to (c, α) .

4. S WORKS

Let $Y \subset X$. Fix α . If there is a 1-1 function $h: Y \rightarrow P_{i,\alpha}$ with $y \in h(y)$ for all y , we say Y is separated by $P_{i,\alpha}$. If $|Y \cap B_{c,\alpha}| \leq 1$ for all c , we say Y is diffuse at stage α .

Lemma 3. Fix $c \neq d \in A$. Let α be a countable limit ordinal, and let $\{\delta_n : n < \omega\}$ be an increasing sequence converging to α . For each n , let $x_n = (c, \delta_n)$, $y_n = (d, \delta_n)$. Suppose $T = \{(x_n, y_n) : n < \omega\} \in M = M_{\beta_\alpha}$. Let G be $\mathbf{P}_\alpha$ -generic over M . Then there is an infinite set $E \in M[G]$, $E \subset \omega$, so that $\{x_n : n \in E\} \cup \{y_n : n \in E\}$ is separated by $P_{i,\alpha}$.

Proof. Fix $m < \omega$ and let $D_{T,m} = \{\tau \in \mathbf{P}_\alpha : \text{there are } n_j, j \leq m, \text{ and distinct } u_j, j \leq 2m, \text{ so each } x_{n_j} \in u_j \in \tau(i) \text{ and each } y_{n_j} \in u_{m+j} \in \tau(i)\}$. For each $\sigma \in \mathbf{P}_\alpha$, $C_\sigma = \bigcup \text{range}(\sigma)$ is compact. Hence each C_σ contains at most finitely many x_n 's and y_n 's, and $D_{T,m}$ is dense.

Lemma 4. Suppose $i \in c$, $M = M_{\beta_\alpha}$, and $\{x_n : n < \omega\}, \{y_n : n < \omega\} \in M[G]$ are separated by $P_{i,\alpha}$, where G is $\mathbf{P}_\alpha$ -generic over M . Let H be $\mathbf{Q}_\alpha$ -generic over $M[G]$. Then there is an infinite set $E \in M[G][H]$, $E \subset \omega$, so that $\{x_n : n \in E\}$ is a subset of B_c , but $\{y_n : n \in E\}$ is diffuse at stage α .

Proof. For each $m < \omega$, $\{\tau \in \mathbf{Q}_\alpha : \text{there are } n_j, j \leq m, \text{ distinct } u_j, j \leq 2m, \text{ and distinct } d_j, j \leq m, \text{ so each } x_{n_j} \in u_j \in \tau(c) \text{ and each } y_{n_j} \in u_{m+j} \in \tau(d_j)\}$ is dense.

Lemma 5. Property A holds.

Proof. Let f be a nontrivial autohomeomorphism of S , and suppose, by contradiction, there are $c \neq d, n, m$ with $f[X_{c,n}] \cap X_{d,m}$ uncountable. Since f is determined by its actions on the atoms of S , $f \in M_\beta$ for some β . Let α be a limit, $\beta_\alpha > \beta$, so $J = \{\delta : f(c, \delta, n) = (d, \delta, m)\}$ is cofinal in α . In $M = M_{\beta_\alpha}$, pick an increasing ω -sequence K in J , K converges to α , and for each $\delta \in K$ let $x_\delta = (c, \delta)$ with $f(x_\delta(n)) = z_\delta(m) \in X_{d,m}$. Let $i \in c$. By Lemma 3 there is an infinite subset K_0 of K so that $\{x_\delta : \delta \in K_0\} \cup \{z_\delta : \delta \in K_0\}$ is separated by $P_{i,\alpha}$.

By Lemma 4 there is an infinite subset K_1 of K_0 so that $\{x_\delta : \delta \in K_1\} \subset B_c$, but $\{z_\delta : \delta \in K_1\}$ is diffuse at stage α . By the remarks in §3, if $Z_0 = \{x_\delta : \delta \in K_1\}$ and $Z_1 = \{z_\delta : \delta \in K_1\}$, then Z_0 converges to (c, α) and Z_1 does not converge. Hence $Z_0(n)$ converges to (c, α, n) and $Z_1(m)$ does not converge. But $Z_1(m) = f[Z_0(n)]$, a contradiction.

Lemma 6. Fix i, j . Let α be a countable limit ordinal, and let $\{\delta_n : n < \omega\}$ be an increasing sequence converging to α . For each n , let $x_n = (\{i\}, \delta_n)$, $y_n = (\{j\}, \delta_n)$. Suppose $T = \{(x_n, y_n) : n < \omega\} \in M = M_{\beta_\alpha}$. Let G be $\mathbf{P}_\alpha$ -generic over M . Then there is an infinite element $E \in M[G]$, $E \subset \omega$, so $\{x_n : n \in E\}$ is separated by $P_{i,\alpha}$ and $\{y_n : n \in E\}$ is separated by $P_{j,\alpha}$.

Proof. If $i = j$ we are done by Lemma 3. So assume $i \neq j$. Fix $m < \omega$ and let $D_{T,m} = \{\tau \in \mathbf{P}_\alpha : \text{there are } n_k, k \leq m, \text{ and distinct } u_k, k \leq 2m, \text{ so each } x_{n_k} \in u_k \in \tau(i) \text{ and each } y_{n_k} \in u_{m+k} \in \tau(j)\}$. For each $\sigma \in \mathbf{P}_\alpha$, $C_\sigma = \bigcup \text{range}(\sigma)$ is compact. So each C_σ contains at most finitely many x_n 's and y_n 's, and $D_{T,m}$ is dense.

Lemma 7. Let $c = \{i, j\}$. Let α be a countable limit ordinal, and let $\{\delta_n : n < \omega\}$ be an increasing sequence converging to α . For each n , let $x_n = (\{i\}, \delta_n)$, $y_n = (\{j\}, \delta_n)$. Suppose $\{(x_n, y_n) : n < \omega\} \in M = M_{\beta_\alpha}$. Then (c, α) is the limit of a subsequence of $\{x_n : n < \omega\}$ and the limit of a subsequence of $\{y_n : n < \omega\}$.

Proof. If $i = j$ we are done by Lemma 4. So assume $i \neq j$. Let G be $\mathbf{P}_\alpha$ -generic over M . Working in $M[G]$, by the previous lemma we may assume $\{x_n : n < \omega\}$ is separated by $P_{i,\alpha}$ and $\{y_n : n < \omega\}$ is separated by $P_{j,\alpha}$. For each $m < \omega$, $\{\tau \in \mathbf{Q}_\alpha : \text{there are } n_k, k \leq m, \text{ distinct } u_k, k \leq 2m, \text{ so each } x_{n_k} \in u_k \in \tau(c) \text{ and each } y_{n_k} \in u_{m+k} \in \tau(c)\}$ is dense.

For the rest of this paper, let $Z = \bigcup_{i < \omega} X_{\{i\}}$.

Lemma 8. $Z \cap [X|\alpha + 1 - X|\alpha]$ is dense in $X|\alpha + 2 - X|\alpha$.

Proof. Let $i \in c \in A$. Let $Y = \{(d, \alpha) : d \in A, |d| = 1\}$. Y is contained in no compact subset of $X|\alpha + 1$. Hence for each $m < \omega$, $\{\tau \in \mathbf{P}_{\alpha+1} : \text{there are distinct } u_j, j < m, \text{ with each } u_j \in \tau(i) \text{ and } u_j \cap Y \neq \emptyset\}$ is dense. So, if G is $\mathbf{P}_{\alpha+1}$ -generic over $M_{\beta_{\alpha+2}}$, there is an infinite set $W \subset Y$, $W \in M[G]$, which is separated by $P_{i,\alpha+2}$. By Lemma 4, $(c, \alpha + 1)$ is in the closure of W .

Corollary 9. Suppose β is a countable limit ordinal. Then $Z - X|\beta$ is dense in $X - X|\beta$.

Proof. By induction, using Lemmas 7 and 8, $\{(c, \beta) : c \in A, |c| = 1\}$ is dense in $X - X|\beta$.

Lemma 10. Property B holds.

Proof. Let f be an autohomeomorphism of S . By Corollary 9, it suffices to show that if $f[X_{c,n}] \equiv X_{c,r}$ and $f[X_{d,n}] \equiv X_{d,s}$, where $c = \{i\}$, $d = \{j\}$, then $r = s$. Pick α so that

- (a) $f \in M_{\beta_\alpha}$,
- (b) $f(c, \delta, n) = (c, \delta, r)$ for all $\delta > \alpha$, and
- (c) $f(d, \delta, n) = (d, \delta, s)$ for all $\delta > \alpha$.

Now pick a limit γ such that $\gamma - \alpha$ is infinite, and let $\{\delta_k : k < \omega\}$ be an increasing sequence in M_{β_γ} converging to γ . Let $x_k = (\{i\}, \delta_k)$, $y_k = (\{j\}, \delta_k)$. The sequence $\{(x_k, y_k) : k < \omega\}$ is in M_{β_γ} . Let $e = \{i, j\}$. By Lemma 7, $z = (e, \gamma)$ is the limit of a subsequence of the x_k 's and the limit of a subsequence of the y_k 's. Hence $z(n)$ is the limit of a subsequence of the $x_k(n)$'s and the limit of a subsequence of the $y_k(n)$'s. So $f(z(n))$ is in $X(r)$, $f(z(n))$ is in $X(s)$, and $r = s$.

5. EXTENSION TO PRODUCTS

In this section we indicate how to extend the methods of this paper to products of groups.

Suppose X, Y satisfy Properties A and B, and let Z be the topological sum of ω many copies of X and of ω many copies of Y . Then the group $(\omega!)^2$ is a subgroup of $G(Z)$. Furthermore, using genericity arguments similar to those in §4, we can construct X, Y so that no nontrivial homeomorphism of Z takes an uncountable subset of a copy of X into a copy of Y . Then $(\omega!)^2 = G(Z)$. This technique extends to other products as well. Its most general form gives the following:

Theorem 11. *Assume there are ω_1 cofinal Cohen reals. Fix $w \leq \omega$, and for each $i < w$ let G_i be either the group $\omega!$ or a separable group of size ω_1 (e.g., the free group on ω_1 generators) as in [Ro2]. Then there are spaces X_i so that each $G_i = G(X_i)$ and $\prod_{i < w} G_i = G(\sum_{i < w} (X_i))$.*

BIBLIOGRAPHY

- [DS] A. Dow and P. Simon, *Thin-tall Boolean algebras and their automorphism groups*, preprint.
- [Ro1] J. Roitman, *Superatomic Boolean algebras*, Handbook of Boolean Algebras, Vol. 3 (J. D. Monk and R. Bonnet, eds.), North-Holland, Amsterdam, 1989, pp. 719–740.
- [Ro2] ———, *Uncountable autohomeomorphism groups of thin-tall locally compact scattered spaces*, preprint.

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