LINEAR INDEPENDENCE OF ITERATES OF ENTIRE FUNCTIONS

LUIS BERNAL GONZÁLEZ

(Communicated by Paul S. Muhly)

ABSTRACT. We prove the following result: The set $\{h_n : n = 0, 1, \dots\}$ is a linearly independent sequence of entire functions, where $h_0 = 1$, $h_1 = g_1$, $h_2 = g_1 \circ g_2$, $h_3 = g_1 \circ g_2 \circ g_3$, $\dots$, g_1 is a nonconstant entire function and g_n ($n \geq 2$) are entire functions which are not polynomials of degree ≤ 1 . Our theorem generalizes a previous one about linear independence of iterates.

In [1] it is proved that except for trivial cases a sequence of iterates of entire functions is always linearly independent. As a related result, it is also shown in [1] that the Feigenbaum functional equation:

$$(1) \quad \begin{aligned} f(f(\lambda x)) + \lambda f(x) &= 0 & (-1 \leq x \leq 1); \\ 0 < \lambda = -f(1) < 1; & \quad f(0) = 1, \end{aligned}$$

does not have an entire solution.

In this paper we shall generalize the interesting theorems proved in [1]. For the sake of completeness, we state them. In the following we denote by $\mathcal{F}$ the space of all entire functions, and by $\mathbb{C}$ the complex plane.

Theorem A. *Let f be a nonidentically zero entire function that satisfies the functional equation:*

$$f(f(\lambda z)) + \lambda f(z) = 0$$

for all $z \in \mathbb{C}$, where λ is a fixed nonzero complex number, then either $f(z) = -z$ and λ is arbitrary or $f(z)$ is a constant with $\lambda = -1$.

From Theorem A, the nonexistence of entire solutions of (1) is derived.

Theorem B. *Let f be an entire function that is not a polynomial of degree ≤ 1 . Let $R_f: \mathcal{F} \rightarrow \mathcal{F}$ be defined by $R_f(g) = g \circ f$ for all $g \in \mathcal{F}$. Then R_f has no eigenvalue distinct from 0, 1 and any eigenfunction with eigenvalue 1 is a constant.*

Theorem C. *Let g be a nonconstant entire function and let f be an entire function which is not a polynomial of degree ≤ 1 . Denote the successive iterates*

Received by the editors February 12, 1990.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 30D05; Secondary 30D15.

Key words and phrases. Entire functions, linear independence, iterates, maximum modulus, asymptotic inequalities, composite functions.

of f in the following way: $f_1 = f$, $f_2 = f \circ f_1$, $\dots$, $f_{n+1} = f \circ f_n$ ($n = 1, 2, \dots$), and f_0 will stand for the identity mapping. Then $\{g \circ f_n: n \geq 0\}$ is a linearly independent sequence of entire functions.

The proofs of these results are based upon a classical result of Pólya (see [5; 2, pp. 50–51]) concerning the growth of Nevanlinna's characteristics of composite functions. We prove a more general theorem without using these characteristics.

First, we need several results about the maximum modulus of an entire function. Throughout this paper we use the following notation. If $f, g, h, f_i, g_i, h_i, \dots$ are entire functions, then $F, G, H, F_i, G_i, H_i, \dots$ respectively are their maximum modulus functions, i.e., $F(r) = \max\{|f(z)|: |z| = r\}$ ($r > 0$) and so on. If $f = \sum_{i=1}^n \lambda_i f_i$ ($\lambda_i \in \mathbb{C}$; $i = 1, 2, \dots, n$) and $h = g \circ f$, then, trivially, $F(r) \leq \sum_{i=1}^n |\lambda_i| F_i(r)$ and $H(r) \leq G(F(r))$ for all $r > 0$. For functions defined on large enough $r > 0$, we say that $\varphi(r) < \psi(r)$ asymptotically (asympt.) when there is $r_0 > 0$ with $\varphi(r) < \psi(r)$ for every $r > r_0$.

The proof of the following theorem can be found, for instance, in [3, pp. 80–81; 4, pp. 225–227].

Theorem D. Assume that $f, g \in \mathcal{F}$, $h = g \circ f$, and $f(0) = 0$. Then, there exists $c \in (0, 1)$ independent from f and g such that

$$G(cF(r/2)) \leq H(r) \quad \text{for all } r > 0.$$

Next, we state two previous lemmas.

Lemma 1. Suppose that $f \in \mathcal{F}$, f is not constant, $\alpha > 1$ and $0 < \beta < \alpha$. Then $\beta F(r) < F(\alpha r)$ asympt.

Proof. We obtain $|g(z)| \leq |z|G(R)/R$ if $|z| \leq R$ by applying Schwarz's lemma to $g(z) = f(z) - f(0)$. If $R = \alpha r$, then $F(r) - |f(0)| \leq G(r) \leq (r/R)(F(R) + |f(0)|) = (F(\alpha r) + |f(0)|)/\alpha$. Choose $\varepsilon = (\alpha - \beta)/(2\alpha + 2\beta)$. Then there is $r_0 > 0$ with $|f(0)| < \varepsilon F(r_0)$. Thus $F(\alpha r) \geq ((1 - \varepsilon)\alpha/(1 + \varepsilon))F(r) > \beta F(r)$ for all $r > r_0$. $\square$

Lemma 2. Assume that p is a nonnegative integer, $p \geq 2$, $g_i \in \mathcal{F}$ ($i = 1, 2, \dots, p$), g_i is a nonconstant function ($i = 2, 3, \dots, p$) and $h = g_1 \circ g_2 \circ \dots \circ g_p$. Then, there exists $d \in (0, 1)$ such that

$$H(r) \geq (G_1 \circ G_2 \circ \dots \circ G_p)(dr) \quad \text{asympt.}$$

Proof. Let us prove the lemma for $p = 2$ and put $g(z) = g_1(z + g_2(0))$ and $f(z) = g_2(z) - g_2(0)$. Evidently, $h = g \circ f$, $f(0) = 0$, $F(r) \geq G_2(r) - |g_2(0)|$ ($r > 0$), $G(s) \geq G_1(s - |g_2(0)|)$ ($s > |g_2(0)|$) and $G_2(r) > (2 + (2/c))|g_2(0)|$ asympt., where c is the constant in Theorem D. Then by this theorem we can conclude that

$$G(cF(r/2)) \geq G_1(cF(r/2) - |g_2(0)|) \geq G_1(cG_2(r/2) - (c+1)|g_2(0)|).$$

Hence

$$(2) \quad H(r) \geq G_1((c/2)G_2(r/2)) \quad \text{asympt.}$$

Let $d = c/6$. By applying Lemma 1 to $\alpha = 3/c$, $\beta = 2/c$ we obtain $G_2(r/2) = G_2(\alpha dr) \geq \beta G_2(dr)$ asymp. This inequality and (2) complete the case $p = 2$.

Let us go on by induction. Assume that there is $d_1 \in (0, 1)$ such that

$$(3) \quad Q(r) \geq (G_1 \circ G_2 \circ \cdots \circ G_{p-1})(d_1 r) \quad \text{asymp.},$$

where $q(z) = g_1 \circ g_2 \circ \cdots \circ g_{p-1}$. We have $h = q \circ g_p$, so we can apply the case $p = 2$. There exists $d_2 \in (0, 1)$ such that

$$(4) \quad H(r) \geq Q(G_p(d_2 r)) \quad \text{asymp.}$$

Again apply Lemma 1 with $\alpha = 2/d_1$, $\beta = 1/d_1$ and set $d = d_1 d_2/2$. Then

$$(5) \quad d_1 G_p(d_2 r) = (1/\beta) G_p(\alpha dr) \geq G_p(dr) \quad \text{asymp.}$$

Equations (3)–(5) complete the proof. $\square$

Theorem. Suppose that g_1 is a nonconstant entire function and g_n ($n \geq 2$) are entire functions which are not polynomials of degree ≤ 1 . Define $h_0 = 1$, $h_1 = g_1$, $h_n = h_{n-1} \circ g_n$ ($n \geq 2$). Then $\{h_n; n = 0, 1, 2, \dots\}$ is a linearly independent sequence of entire functions.

Proof. By contradiction, let p be an integer ≥ 2 and let $\sum_{i=0}^p \lambda_i h_i(z) = 0$ for all $z \in \mathbb{C}$, with $\lambda_i \in \mathbb{C}$ ($i = 0, 1, \dots, p$) and $\lambda_p \neq 0$ (cases $p = 0$ and $p = 1$ are trivial). Then $h_p = \sum_{i=0}^{p-1} \mu_i h_i$, where $\mu_i = -\lambda_i/\lambda_p$ ($i = 0, 1, \dots, p-1$). We have $H_p(r) < \mu \cdot \sum_{i=0}^{p-1} H_i(r)$ ($r > 0$), with μ a fixed positive real number greater than $\max\{|\mu_i|; i = 0, 1, \dots, p-1\}$. By applying Lemma 2, we have $(G_1 \circ G_2 \circ \cdots \circ G_p)(dr) < \mu + \mu \sum_{i=1}^{p-1} (G_1 \circ G_2 \circ \cdots \circ G_i)(r)$ asymp. Define φ by

$$\varphi(r) = (G_1 \circ G_2 \circ \cdots \circ G_{p-1})(r) \quad (r > 0).$$

Evidently, $1 < \varphi(r)$ asymp. and $(G_1 \circ G_2 \circ \cdots \circ G_i)(r) \leq \varphi(r)$ asymp. ($i = 1, 2, \dots, p-1$), because $G_j(r) > r$ asymp. for all $j \geq 2$ and each G_j ($j \geq 1$) is strictly increasing, by hypothesis. Hence,

$$(6) \quad \varphi(G_p(dr)) < p\mu\varphi(r) \quad \text{asymp.}$$

Since g_p is not a polynomial of degree ≤ 1 , the inequality

$$G_p(r) > (p/d)(1 + \mu)r \quad \text{asymp.}$$

holds. Then we apply Lemma 1 $p-1$ times to obtain $\varphi(G_p(dr)) > \varphi((p + p\mu)r) > (G_1 \circ G_2 \circ \cdots \circ G_{p-2})((p-1 + p\mu)G_{p-1}(r)) > (G_1 \circ \cdots \circ G_{p-3})((p-2 + p\mu)(G_{p-2} \circ G_{p-1})(r)) > \cdots > (1 + p\mu)(G_1 \circ \cdots \circ G_{p-1})(r) > p\mu\varphi(r)$ asymp., that is a contradiction with (6). $\square$

Finally, we derive Theorems A, B, and C as a corollary.

Proof of Theorem A. If f is a polynomial of degree ≤ 1 , then the statement is obvious. Now, let f be an entire function that is not a polynomial of degree ≤ 1 . Then f does not satisfy $f(f(\lambda z)) + \lambda f(z) = 0$ ($z \in \mathbb{C}$) for any nonzero

$\lambda \in \mathbb{C}$. To see this, we only have to apply our theorem to $g_1 = f$, $g_n = f(\lambda z) (n \geq 2)$.

Proof of Theorem B. Let g be an eigenfunction for R_f . Applying our theorem to $g_1 = g$, $g_n = f$ ($n \geq 2$), we deduce that g is a constant ($g(z) = \mu \neq 0$). Consequently, there is $\lambda \in \mathbb{C}$ such that $\mu - \lambda\mu = 0$. Thus, in fact, $\lambda = 1$ is the unique eigenvalue of R_f and the eigenfunctions are precisely the set of all nonzero constant entire functions.

Proof of Theorem C. Again apply our theorem to $g_1 = g$, $g_n = f$ ($n \geq 2$).

REFERENCES

1. J.P.R. Christensen and P. Fischer, *Linear independence of iterates and entire solutions of functional equations*, Proc. Amer. Math. Soc. **103** (1988), 1120–1124.
2. W. K. Hayman, *Meromorphic functions*, Oxford Mathematical Monograph, Clarendon Press, 1964.
3. A. S. B. Holland, *Introduction to the theory of entire functions*, Academic Press, New York and London, 1973.
4. J. E. Littlewood, *Lectures on the theory of functions*, Oxford Univ. Press, London and New York, 1944.
5. G. Pólya, *On an integral function of an integral function*, J. London Math. Soc. **1** (1926), 12–15.

DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, FACULTAD DE MATEMÁTICAS, AVENIDA REINA MERCEDES, 41080-SEVILLA, SPAIN