ON THE RELATIONSHIP BETWEEN DENSITY AND WEAK DENSITY IN BOOLEAN ALGEBRAS

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ABSTRACT. Given a homogeneous, complete Boolean algebra B , it is shown that $d(B) \leq \min(2^{<wd(B)}, \sup\{\lambda^{c(B)} : \lambda < wd(B)\})$ in ZFC, where $d(B)$ is the density, $wd(B)$ is the weak density, and $c(B)$ is the cellularity of B . A corollary to this result is that $d(B) = wd(B)$ in $ZFC + GCH$.

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper, B , or more properly $(B, +, \cdot, -, 0, 1)$, denotes a homogeneous, complete Boolean algebra and $|X|$ denotes the cardinality of a set X . A subset X of B is *dense* if and only if $\forall a \in B \setminus \{0\}, \exists$ nonzero $x \in X$ such that $x \leq a$. A subset X of B is *weakly dense* if and only if $\forall a \in B \setminus \{0\}, \exists$ nonzero $x \in X$ such that $x \leq a$ or $x \leq -a$. The *density* of B is defined by $d(B) = \min\{|X| : X \text{ is dense}\}$. The *weak density* of B is defined by $wd(B) = \min\{|X| : X \text{ is weakly dense}\}$. Clearly, $wd(B) \leq d(B)$. Recently, there has been interest in determining a bound for $d(B)$ in terms of $wd(B)$. M. Burke has shown $wd(B) = d(B)$ for measure algebras [2]. Recall that X is an *antichain* means $X \subset B \setminus \{0\}$ and consists of pairwise disjoint elements and the *cellularity* of B , $c(B) = \sup\{|A| : A \text{ is an antichain}\}$ [4]. W. Just has shown that $wd(B) = d(B)$ in $ZFC + GCH$ for c.c.c. algebras (i.e., algebras for which $c(B)$ is countable) [3]. We will establish that

$$d(B) \leq \min(2^{<wd(B)}, \sup\{\lambda^{c(B)} : \lambda < wd(B)\})$$

in ZFC. It immediately follows from this inequality that $wd(B) = d(B)$ in $ZFC + GCH$.

Concerning notation and terminology, X will always indicate a subset of B , $X^+ = X \setminus \{0\}$, $X|a = \{x \cdot a : x \in X\}$, and a and b are *disjoint* means $a \neq b$ and $a \cdot b = 0$. X is said to be *homogeneously weakly dense* if and only if for each a in B^+ , $X|a$ is weakly dense in $B|a$. The *homogeneous weak density*

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of B , $\text{hwd}(B) = \min\{|X|: X \text{ is homogeneously weakly dense}\}$. X is *nowhere relatively dense* if and only if for each a in B^+ , $X|a$ is not dense in $B|a$. The *concentration* of X at a is defined by $\text{con}(X, a) = \{x \in X: x \cdot a \neq 0\}$. The *subalgebra* generated by X is denoted by $\mathcal{A}(X)$. For a function $f: G \rightarrow H$, $f''(G) = \{f(g): g \in G\}$. Throughout, we use standard set-theoretic notation, e.g., ω denotes the set of nonnegative integers.

Observation 1.1. X is weakly dense if and only if for all a in B $\text{con}(X, a) \neq \text{con}(X, -a)$.

Observation 1.2. X is homogeneously weakly dense if and only if whenever a and b are disjoint, $\text{con}(X, a) \neq \text{con}(X, b)$.

Proof. If X is homogeneously weakly dense and a and b are disjoint, then $X|(a+b)$ is weakly dense in $B|(a+b)$ in which case $\text{con}(X, a) \neq \text{con}(X, b)$, by Observation 1.1. Conversely, suppose $\forall$ disjoint $a, b \in B$, $\text{con}(X, a) \neq \text{con}(X, b)$. Let $c \in B^+$. Then $\forall d \in B$, $\text{con}(X, c \cdot d) \neq \text{con}(X, c \cdot (-d))$. By Observation 1.1, $X|c$ is weakly dense in $B|c$. Thus, X is homogeneously weakly dense. $\square$

Observation 1.3. X is nowhere relatively dense if and only if for each a in B^+ , there is some nonzero $d < a$ such that $\text{con}(X, d) \subset \text{con}(X, a - d)$.

2. HOMOGENEOUS WEAK DENSITY

If a subset is nowhere relatively dense, is it “sparse” enough to exclude the condition of being weakly dense? In other words, if a subset X is weakly dense does there necessarily exist a in B^+ such that $X|a$ is dense in $B|a$? The answer is “no.” If one considers collapsing algebras there may be constructed a weakly dense set which is at the same time nowhere relatively dense [1].

In this section, we answer a similar question concerning the notion of homogeneously weakly dense. The condition of being homogeneously weakly dense is strictly stronger than that of being weakly dense. For example, let $a \in B \setminus \{1, 0\}$. Note that $B|a$ is weakly dense in B but not homogeneously weakly dense in B . However, the following theorem does hold.

Theorem 2.1. Let X be weakly dense. There exists $c \in B^+$ such that $X|c$ is homogeneously weakly dense in $B|c$. Consequently, $\text{hwd}(B) = \text{wd}(B)$.

Proof. We assume $X|c$ is not homogeneously weakly dense in $B|c$ for all $c \in B^+$. Then there is a dense set of $a \in B^+$ such that $X|a$ is not weakly dense in $B|a$. Hence there is a maximal antichain of such a . Now for each $a \in A$ write $a = a_0 + a_1$ where $a_0 \cdot a_1 = 0$ and $\text{con}(X, a_0) = \text{con}(X, a_1)$. Then $\sum_{a \in A} a_0$ and $\sum_{a \in A} a_1$ both meet every member of X , contradicting the weak density of X . The fact that $\text{hwd}(B) = \text{wd}(B)$ follows from the homogeneity of B . $\square$

3. MAIN RESULT

Throughout this section, let $X = (x_\tau)_{\tau < \kappa}$ be a homogeneously weakly dense set with $|X| = \text{wd}(B) = \kappa$. For each $\rho < \kappa$, Y_ρ denotes $(x_\tau)_{\tau \leq \rho}$. W is a

bounded subset of X means $\exists \rho < \kappa$ such that $W \subset Y_\rho$. Set $E = \{\Sigma W : W \text{ is a bounded subset of } X\}$ and $D = \mathcal{A}(E)$, the subalgebra generated by E . Observe that if W is a subset of X , it has a subset of size at most $c(B)$ with the same supremum. Hence, $|D| \leq \min(2^{<\text{wd}(B)}, \sup\{\lambda^{c(B)} : \lambda < \text{wd}(B)\})$.

Theorem 3.1. *There exists $a \in B^+$ such that $D|a$ is dense in $B|a$. Consequently, $\text{d}(B) \leq \min(2^{<\text{wd}(B)}, \sup\{\lambda^{c(B)} : \lambda < \text{wd}(B)\})$.*

Proof. By way of contradiction, we will assume D is nowhere relatively dense. Based on this assumption, we prove two lemmas.

Lemma 3.2. *For each $\rho < \kappa$ and $a \in B^+$, there exist nonzero $c < a$, an antichain A in $B|(a - c)$, and a function $f : A \rightarrow \kappa$ such that*

- (i) $\forall p \in \text{con}(X, c), \sup(f''(\text{con}(A, p))) = \kappa$,
- (ii) $\text{con}(Y_\rho, c) = \text{con}(Y_\rho, \Sigma A)$, and
- (iii) $\Sigma A = a - c$.

Proof of the lemma. Fix $\rho < \kappa$ and $a \in B^+$. We will first show that $\exists c, A$, and f such that conditions (i) and (ii) are satisfied. Since D is nowhere relatively dense, by Observation 1.3, $\exists$ nonzero $s < a$ such that $\text{con}(D, s) \subset \text{con}(D, a - s)$. Let $U = \{x \in X : x \cdot s = 0\}$ and $u = a \cdot \Sigma U$. Observe that $s \leq (-u) \cdot a$. Also note that if $x \in U$, $x \notin \text{con}(X, (-u) \cdot a)$. Thus, we have $\text{con}(X, (-u) \cdot a - s) \subset \text{con}(X, s)$. By the homogeneous weak density of X , $\exists q \in X$ such that $0 < q \cdot (-u) \cdot a \leq s$. Set $c = q \cdot (-u) \cdot a$, $T = \{x \in X : x \cdot c = 0\}$ and $t = a \cdot \Sigma T$. For each $\alpha < \kappa$, set $T_\alpha = Y_\alpha \cap T$ and $t_\alpha = a \cdot \Sigma T_\alpha$ (if $T_\alpha = \emptyset$ set $t_\alpha = 0$). Then $\forall \alpha < \kappa$, $t_\alpha \in D|a$. Also, observe that since $c \leq s$, we have $u \leq t$ and $(-t) \cdot a \leq -u \cdot a$. Set

$$A = \left\{ t_\alpha - \sum_{\tau < \alpha} t_\tau : \alpha > \rho \text{ and } t_\alpha - \sum_{\tau < \alpha} t_\tau \neq 0 \right\}.$$

Define $f : A \rightarrow \kappa$ by $f(t_\alpha - \sum_{\tau < \alpha} t_\tau) = \alpha$. We show (i) by way of contradiction. Suppose $\exists p \in \text{con}(X, c)$ such that $\sup(f''(\text{con}(A, p))) < \kappa$. Then $\exists \alpha < \kappa$ such that $p \cdot t = p \cdot t_\alpha$. We have $p \cdot q \cdot a = p \cdot q \cdot t + p \cdot q \cdot a \cdot (-t) \leq p \cdot q \cdot t_\alpha + p \cdot q \cdot a \cdot (-u) \leq p \cdot q \cdot t_\alpha + s$. Thus, setting $w = p \cdot q \cdot a \cdot (-t_\alpha)$, we have $w \in D|a$ and $0 < w \leq s$. This contradicts the fact that $\text{con}(D, s) \subset \text{con}(D, a - s)$. To show (ii) we need to show $\text{con}(Y_\rho, \Sigma A) \subset \text{con}(Y_\rho, c)$. Suppose $x_\pi \in \text{con}(Y_\rho, \Sigma A)$. Then $\exists \alpha > \rho \geq \pi$ such that $x_\pi \cdot (t_\alpha - \sum_{\tau < \alpha} t_\tau) \neq 0$ which implies $x_\pi \notin T_\pi$, i.e., $x_\pi \cdot c \neq 0$. Thus, $x_\pi \in \text{con}(Y_\rho, c)$.

Let $\mathbf{P}$ consist of all triples (c, A, f) such that (i) and (ii) are satisfied. (For convenience, $h \in \mathbf{P}$ may be expressed as $h = (\pi_1(h), \pi_2(h), \pi_3(h))$. We have shown that $\mathbf{P}$ is nonempty. Partial order $\mathbf{P}$ by $(c, A, f) \leq^* (c_1, A_1, f_1)$ if and only if $c \leq c_1$, and $\forall t \in A$, $\exists t_1 \in A_1$ such that $t \leq t_1$ and $f(t) = f_1(t_1)$.

If Q is a chain in $(\mathbf{P}, \leq^*)$, set $c_M = \sum_{q \in Q} \pi_1(q)$. $\forall \alpha < \kappa$, set $h_\alpha = \sum \{\Sigma f^{-1}(\alpha) : f = \pi_3(q) \text{ for some } q \in Q\}$. (Take $h_\alpha = 0$ if the preceding set is empty.) Setting $A_M = (h_\alpha)_{\alpha < \kappa}$, we have A_M^+ is an antichain. Define the

function $f_M: A_M^+ \rightarrow \kappa$ by $f_M(h_\alpha) = \alpha$. Clearly, (c_M, A_M^+, f_M) is a bound for Q .

Let (c, A, f) be a maximal element of $\mathbf{P}$. We will show that (iii) is satisfied. If this were not so, then $0 \neq a_1 = a - (c + \Sigma A)$. Choose nonzero $c_1 < a_1$, an antichain A_1 in $B|(a_1 - c_1)$, and a function $f_1: A_1 \rightarrow \kappa$ such that (i) $\forall p \in \text{con}(X, c_1)$, $\sup(f_1''(\text{con}(A_1, p))) = \kappa$ and (ii) $\text{con}(Y_p, c_1) = \text{con}(Y_p, \Sigma A_1)$. Then $(c, A, f) <^* (c + c_1, A \cup A_1, f \cup f_1)$. This contradicts maximality and completes the proof of the lemma. $\square$

Lemma 3.3. *There exists an antichain $(c_n)_{n < \omega}$ such that $[\text{con}(X, c_n)]_{n < \omega}$ is nondecreasing.*

Proof of the lemma. For each $c \in B^+$ we define the property $\sigma(c)$ as follows: $\sigma(c)$: There exist an antichain A in $B|(-c)$ and a function $f: A \rightarrow \kappa$ such that for each x in $\text{con}(X, c)$, $\sup(f''(\text{con}(A, x))) = \kappa$.

Applying Lemma 3.2 to the element 1, we know that there exists $c_0 \in B^+$ such that $\sigma(c_0)$ holds. For each n , $0 < n < \omega$, we will choose $c_n \in B^+$ such that the following conditions called $\varphi(n)$ hold:

- (1) $(\sum_{k < n} c_k) \cdot c_n = 0$,
- (2) if $x \in X$ and $x \cdot (\sum_{k < n} c_k) \neq 0$, then $x \cdot c_n \neq 0$, and
- (3) $\sigma(\sum_{k \leq n} c_k)$.

Fix n , $0 < n < \omega$. Assume $(c_k)_{k < n}$ have been constructed. By condition (3) of $\varphi(n-1)$, $\sigma(\sum_{k < n} c_k)$ holds (if $n = 1$ the preceding expression is merely $\sigma(c_0)$). There is an antichain Z in $B|(-\sum_{k < n} c_k)$ and a function $v: Z \rightarrow \kappa$ such that if $x \in X$ and $x \cdot (\sum_{k < n} c_k) \neq 0$, then $\sup(v''(\text{con}(Z, x))) = \kappa$. Using our Lemma 3.2, for each $z \in Z$ we choose nonzero $s_z < z$, an antichain K_z in $B|(z - s_z)$, and a function $r_z: K_z \rightarrow \kappa$ such that

- (i) $\forall p \in \text{con}(X, s_z)$, $\sup(r_z''(\text{con}(K_z, p))) = \kappa$,
- (ii) $\text{con}(Y_{v(z)}, s_z) = \text{con}(Y_{v(z)}, \Sigma K_z)$, and
- (iii) $\Sigma K_z = z - s_z$.

Observation. Let $\tau < \kappa$ and $z \in Z$. If $v(z) \geq \tau$ and $x_\tau \cdot z \neq 0$, then $x_\tau \cdot s_z \neq 0$. (This follows directly from (ii) and (iii).)

Set $c_n = \sum_{z \in Z} s_z$. Clearly, condition (1) of $\varphi(n)$ is satisfied. Consider $x_\tau \in X$ such that $x_\tau \cdot (\sum_{k < n} c_k) \neq 0$. By assumption, there is some element $z_0 \in Z$ such that $x_\tau \cdot z_0 \neq 0$ and $v(z_0) \geq \tau$. By the above observation, we have $x_\tau \cdot s_{z_0} \neq 0$ which implies $x_\tau \cdot c_n \neq 0$. Thus, condition (2) of $\varphi(n)$ holds. Set $A = \bigcup_{z \in Z} K_z$ and $f = \bigcup_{z \in Z} r_z$. Then A is an antichain and f is a function from A into κ . Suppose $x_\tau \cdot (\sum_{k \leq n} c_k) \neq 0$. To demonstrate condition (3) of $\varphi(n)$ we first show that there exists $z_1 \in Z$ such that $x_\tau \cdot s_{z_1} \neq 0$. Note that $(\sum_{k \leq n} c_k) = \sum_{z \in Z} s_z + (\sum_{k < n} c_k)$. If $x_\tau \cdot \sum_{z \in Z} s_z \neq 0$, then certainly there exists $z_1 \in Z$ such that $x_\tau \cdot s_{z_1} \neq 0$. Otherwise, $x_\tau \cdot (\sum_{k < n} c_k) \neq 0$, and by assumption,

there exists $z_1 \in Z$ such that $x_\tau \cdot z_1 \neq 0$ and $v(z_1) \geq \tau$, in which case, $x_\tau \cdot s_{z_1} \neq 0$. Observe that $r''_{z_1}(\text{con}(K_{z_1}, x_\tau)) = f''(\text{con}(K_{z_1}, x_\tau)) \subset f''(\text{con}(A, x_\tau))$. Since $x_\tau \in \text{con}(X, s_{z_1})$, we have $\sup(f''(\text{con}(A, x_\tau))) \geq \sup(r''_{z_1}(\text{con}(K_{z_1}, x_\tau))) = \kappa$. We have shown condition (3) of $\varphi(n)$ is satisfied. This completes the proof of the lemma. $\square$

To complete the proof of Theorem 3.1, let $(c_n)_{n < \omega}$ be as in Lemma 3.3. Then $\text{con}(X, \sum_{n < \omega} c_{2n}) = \text{con}(X, \sum_{n < \omega} c_{2n+1})$. But, this contradicts the fact that X is homogeneously weakly dense. Our assumption that D is nowhere relatively dense is wrong. Finally,

$$d(B) \leq \min(2^{<\text{wd}(B)}, \sup\{\lambda^{c(B)} : \lambda < \text{wd}(B)\})$$

follows from homogeneity. $\square$

Remark. There are situations for which $2^{<\text{wd}(B)}$ is less than

$$\sup\{\lambda^{c(B)} : \lambda < \text{wd}(B)\}$$

and vice versa. For example, if $\text{wd}(B) = c(B) = \omega_1$, then CH implies $2^{<\text{wd}(B)} = \omega_1$; however, $\sup\{\lambda^{c(B)} : \lambda < \text{wd}(B)\} \geq \omega_2$. Also, if B is c.c.c. and $\text{wd}(B) = \omega_\omega$, then $2^\omega < \omega_\omega$ implies $\sup\{\lambda^{c(B)} : \lambda < \text{wd}(B)\} = \omega_\omega$ regardless of how big 2^{ω_n} ($0 < n < \omega$) are.

Corollary 3.4. $d(B) = \text{wd}(B)$ in ZFC + GCH.

Proof. For all infinite cardinals κ , $\kappa = 2^{<\kappa}$ in ZFC + GCH. $\square$

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