PROJECTIVELY NORMAL ADJUNCTION SURFACES

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ABSTRACT. On a projective surface S polarized by a very ample line bundle L one can consider the adjoint bundles $(K_S \otimes L)^{\otimes n} = L^{\otimes n}$ and the adjunction mappings associated to them. Suppose these mappings are embeddings (it is well known when this is the case: see [So-VdV]). We prove that these embeddings are projectively normal for $n \geq 2$ and we describe some counterexamples for $n = 1$. For $n \geq 2$ we show that the ideals of the image of S are generated by quadrics.

Let S' be a projective surface polarized by a very ample line bundle L' and let $L' = K_{S'} + L'$ be the *adjoint bundle*. Let $r: S' \rightarrow S$ be the contraction of all the rational curves $P \subset S'$, such that $L' \cdot P = -K_{S'} \cdot P = 1$, to a finite number of smooth points in a new surface S . Of course the map associated to $\Gamma(L')$ will factor through r , and $L' = r^*(K_S + L) = r^*(L)$, where L is an ample line bundle on S such that $L = r_*(L')$. In [So-VdV] it is proved that if (S', L') is not in a well-known list of polarized pairs then the map associated to $\Gamma(L)$ is an embedding (we just mention that this paper concludes the study of the very ampleness of the bundle L , using the technique of Reider, which was a very old question going back to Castelnuovo and Enriques (see also [Se1]).

It is natural at this point to ask for the projective normality of this embedding (we will say L is projectively normal). If there is a smooth hyperelliptic curve C in the linear system $|L|$, then, by the Noether theorem and the adjunction formula, L is not projectively normal (precisely L is not 1- and 2-normal); but even if there is no such element in $|L|$ we will show an example for which the projective normality of L fails.

However we prove that $L^{\otimes n}$ is projectively normal for all $n \geq 2$, and that the ideal of the image of the surface S through the map associated at $\Gamma(L^{\otimes n})$ is generated by quadrics.

Moreover, if $R(L, S) =: \bigoplus_m H^0(O_S(mL)) = \bigoplus_m R_m$ is the graded ring associated to L , then $r(L, S)$, the maximal degree of the elements of $R(L, S)$ which appears in a minimal system of homogeneous generator, is less than or equal to two.

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The proofs rely on the theory of the Koszul cohomology developed by Mark L. Green [Gr].

1. NOTATION AND PRELIMINARIES

1.1. In this paper we consider projective surfaces S , which means that S is a smooth, irreducible, projective scheme of dimension 2, defined over the field of complex numbers.

If D is a divisor on a surface S we will use the following notation:

- $\mathcal{O}_S(D) =:$ the invertible sheaf associated to D ,
- $h^i \mathcal{O}_S(D) =: \dim H^i(\mathcal{O}_S(D))$,
- $\chi(\mathcal{O}_S(D)) =: \sum_{i=0,1,2} (-1)^i h^i \mathcal{O}_S(D)$,
- $K_S =:$ the canonical divisor on S ,
- $g(D) =: \frac{1}{2}(D \cdot D + D \cdot K_S) + 1$, the arithmetic genus of D ,
- $p_g(S) =: \dim H^0(\mathcal{O}_S)$, the geometric genus of S ,
- $q(S) =: \dim H^1(\mathcal{O}_S)$, the irregularity of S ,
- $k(S) =: \operatorname{tr} \deg(\bigoplus_{n=0}^{\infty} H^0(S, \mathcal{O}_S(nK_S))) - 1$, the Kodaira dimension of S .

We do not distinguish notationally between a divisor and its associated line bundle.

1.2. According to what we have said in the introduction and to what we are going to prove, from now on we will consider a line bundle L on S which is at least nef and big and such that $L = K_S + L$ is spanned by global sections and big.

1.3. We can associate to L a graded ring

$$R(L, S) =: \bigoplus_m H^0(\mathcal{O}_S(L^{\otimes m})) = \bigoplus_m R_m,$$

and we define the L -dimension of S by

$$k(S, L) := \operatorname{tr} \deg R(L, S) - 1.$$

From our assumption 1.2 we have that $k(S, L) = 2$ and, by a theorem of Mumford [Mu], that $R(L, S)$ is finitely generated.

Finally $r(L, S)$ will denote the maximal degree of the elements of $R(L, S)$ which appears in a minimal system of homogeneous generators.

1.4. If we call $\varphi_{nL}: S \rightarrow \mathbf{P}^N$ the map associated to $L^{\otimes n} := H$ for $n \geq 1$, we have the following

Definition. H is said to be projectively normal if for every $\rho \geq 0$ the maps

$$H^0(\mathbf{P}^N, \mathcal{O}(\rho)) \cong S^\rho H^0(S, H) \rightarrow H^0(S, H^\rho)$$

are surjective.

1.4.1. *Remark.* In the case $p = 0$ the maps are surjective for all $n \geq 1$; in the case $p = 1$ they are surjective if H is very ample.

2. THE KOSZUL COHOMOLOGY AND THE PROJECTIVE NORMALITY OF $L^{\otimes n}$

As mentioned in the introduction, we will obtain our results via the theory of the Koszul cohomology as developed by Mark L. Green in [Gr], which is applied to our particular case, that is, on a projective surface S for the line bundle $L = K_S + L$ with L nef and big. We constantly refer directly to [Gr].

2.1. We briefly recall that in the notation of 1.3, that is, with $R_m = H^0(S, L^{\otimes m})$, we can construct the *Koszul complex*

$$\cdots \rightarrow \Lambda^{p+1} R_1 \otimes R_{q-1} \xrightarrow{d_{p+1, q-1}} \Lambda^p R_1 \otimes R_q \xrightarrow{d_{p, q}} \Lambda^{p-1} R_1 \otimes R_{q+1} \rightarrow \cdots$$

and the *Koszul cohomology groups* are defined by

$$\mathcal{K}_{p, q}(S, L) := \text{Ker } d_{p, q} / \text{Im } d_{p, q}.$$

We now prove the following

2.2. Proposition. *Let S , L , and L be as in 1.2. Then $\mathcal{K}_{p, q}(S, L) = 0$ for all $q \geq 4$ and $p \leq p_g(S) - q(S) + g(L) - 3$ and for $q = 3$ and $p \leq g(L) - q(S) - 3$.*

Proof. By the Kodaira vanishing theorem and the Serre duality we have

$$H^1(S, L^{\otimes p}) = 0 \quad \text{for } p \neq 0;$$

hence for $q \geq 3$, we are in the hypothesis of the “duality theorem,” that is, the theorem 2.c.6 in [Gr].

This tells us that

$$(\mathcal{K}_{p, q}(S, L))^* = \mathcal{K}_{r-2-p, 3-q}(S, K_S, L) \quad \text{with } r = h^0(L) - 1.$$

On the other hand the “vanishing theorem,” that is, the Theorem 3.a.1 in [Gr], asserts that

$$\mathcal{K}_{r-2-p, 3-q}(S, K_S, L) = 0 \quad \text{if } h^0(S, K_S - (q-3)L) \leq r - 2 - p.$$

If $q \geq 4$ then $h^0(S, K_S - (q-3)L) = 0$, by the hypothesis, and therefore we are asking for

$$p \leq r - 2 = h^0(L) - 3 = p_g(S) - q(S) + g(L) - 3.$$

If $q = 3$ then $h^0(S, K_S - (q-3)L) = p_g(S)$, and we are asking for

$$p \leq g(L) - q(S) - 3.$$

2.3. Corollary. *In the hypothesis of the proposition and if $h^0(L) \geq 4$, we have*

$$R_m = R_1 R_{m-1} \quad \text{for all } m \geq 3.$$

Proof. By the construction of the Koszul cohomology and since $\Lambda^p R_1 = 0$ for $p < 0$, we have that our corollary will follow immediately if we prove that $\mathcal{K}_{0, q}(S, L) = 0$ for all $q \geq 3$.

Since, by hypothesis, we can assume that $p_g(S) - q(S) + g(L) = h^0(L) \geq 3$ we have immediately from the proposition that $\mathcal{K}_{0,q}(S, L) = 0$ for all $q \geq 4$.

On the other hand $g(L) - q(S) - 3 = h^0(L) - h^0(K_S) - 3 \geq 0$: this is true as above if $h^0(K_S) = 0$ and by the Hopf lemma otherwise (i.e., $h^0(L) - h^0(K_S) = h^0(K_S + L) - h^0(K_S) \geq h^0(L) - 1 \geq 3$). This will give $\mathcal{K}_{0,q}(S, L) = 0$ for $q = 3$.

As a consequence of the corollary we have the following

2.4. Theorem. *Let S be a projective surface, and L a nef and big line bundle on S such that $h^0(L) \geq 4$ and $L = K_S + L$ is spanned by global sections and big.*

Then

- (a) $r(S, L) \leq 2$,
- (b) for every $n \geq 2$, if $L^{\otimes n}$ is very ample then it is projectively normal.

If moreover $h^0(L) \geq 4$, then

- (c) the ideal of $\varphi_{nL}(S)$ is generated by quadrics.

Proof. (a) and (b) follow immediately from Corollary 2.3. (c) on the other hand follows from $\mathcal{K}_{1,q}(S, L) = 0$ for all $q \geq 3$; this is proved as in the case $p = 0$ with the hypothesis $h^0(L) \geq 4$.

2.5. Remark. As pointed out in the introduction, consider a polarized pair (S', L') with L' very ample and which is neither $(\mathbb{P}^2, \mathcal{O}(e))$ with $e = 1$ or 2 , nor a quadric, nor a scroll, nor a conic bundle, nor a Del-Pezzo surface; then $L' = K_{S'} + L'$ is spanned by global sections and big and the map associated to $L'^{\otimes n}$ for $n \geq 2$, after contracting the (-1) -lines relative to L , is an embedding (see [So-VdV]). By our theorem this embedding is projectively normal and the ideal of the image is generated by quadrics.

3. THE ADJOINT BUNDLE L

Assuming that L is very ample, it is natural at this point to ask for the projective normality of L itself.

By the previous result this is equivalent at the surjectivity of the map

$$S^2 H^0(S, L) \longrightarrow H^0(S, L^{\otimes 2})$$

(the other being surjective by Corollary 2.3).

Unfortunately this map is not surjective in many cases: for instance, as mentioned in the introduction, that is the case if $|L|$ contains a smooth hyperelliptic element, by the adjunction formula, the Noether theorem, and the Kodaira vanishing theorem.

On the other hand the map is not surjective as soon as

$$\dim H^0(S, L^{\otimes 2}) > \dim S^2 H^0(S, L),$$

or equivalently

$$(p_g(S) - q(S) + g(L))^2 < p_g(S) + 3g(L) - q(S) - 2 + 2d'$$

with $d' = L \cdot L$.

Polarized surfaces (S, L) with L very ample, with no smooth hyperelliptic element in $|L|$ and satisfying this inequality, and therefore not projectively normal relatively to L , exist: for instance the abelian surfaces of degree 10 in $\mathbf{P}^4$.

Another surface of this type is the following: let S be a minimal bielliptic surface in $\mathbf{P}^4$, i.e., $S \subset \mathbf{P}^4$ is a minimal surface with $k(S) = 0$, $q(S) = 1$, $p_g(S) = 0$. F. Serrano proved recently the existence of such a surface and that for it we have $L \cdot L = L \cdot L = 10$, and $g(L) = 6$ [Se2].

4. FURTHER REMARKS

In this section we describe one of our initial motivations for the construction of as many projectively normal surfaces $S \subset \mathbf{P}^N$ as possible; the reader will find no theorems here. We fix one such surface and we assume moreover that $h^i(\mathcal{O}_S(1)) = 0$ for $i = 1, 2$ (as in the cases $|\mathcal{O}_S(1)| = |2K + 2L|$, etc.). Calling $H = \mathcal{O}_S(1)$ we let $d = \deg S = H \cdot H$ and $g = g(H)$.

Now take a general hyperplane $M \subset \mathbf{P}^N$ and set $C := S \cap M$. We have $h^1(\mathcal{O}_C(1)) = p_g$ and $h^1(I_{C,M}(1)) = q$; $H^0(M, \mathcal{O}_M(t)) \rightarrow H^0(C, \mathcal{O}_C(t))$ are surjective for every $t \geq 2$ and in particular C has maximal rank. We think that the curves obtained in this way are very interesting.

First of all, using the Euler sequence, we get $h^1(N_{C/M}) \geq p_g N - (4g - 2)$. Thus very often $h^1(N_{C/M})$ is huge; for instance if $|H| = |2K + 2L|$, this h^1 is always $\neq 0$ if $p_g \geq 8$. The standard degeneration methods to obtain curves of maximal rank [Ba-El 1; Ba-El 2] show only the existence of curves, D , with maximal rank and with $h^1(N_{D/M}) = 0$. Furthermore, by construction, these curves C are in the unknown part of M_g which is the union of rational curves.

Now fix an irreducible component W of $\text{Hilb}(\mathbf{P}^{N-1})_{\text{red}}$, W parametrizing (part of the) curves of given degree d and arithmetical genus g , and let $W' \subset W$ the part parametrizing curve with at most one node as singularities (alternatively: any number of nodes allowed) and with maximal rank. Assume $N \geq 6$. Fix a general point $x \in \mathbf{P}^N$ such that $x \notin \text{Sec}(S)$ and also a (general) Lefschetz pencil A with a $\mathbf{P}^{N-2}$ not containing x as axis. By the generality of M and the pencil, we may take M to be an element of the pencil. We assume, since $M = \mathbf{P}^{N-1}$, that $C \in W$ (hence $C \in W'$). Projecting from x to M we find a pencil of smooth curves in W . They are all nondegenerate and with maximal rank, except the one corresponding to the hyperplane of A containing x . Thus we get an injective morphism from the affine line to W' .

Over $\mathbf{C}$ the existence of nonconstant holomorphic maps $\mathbf{C} \rightarrow W'$ gives mild restrictions on the complex function theory of W' .

To obtain more, we assume from now on that (as in Theorem 2.4) the homogeneous ideal of S is generated by quadrics. Fix a point $P \in C$. Note that the assumption on the homogeneous ideal of S implies that for every line $D \subset M$ through P , $\text{length}(D \cap C) \leq 2$. This means exactly that the image C' of C under the projection from P into a $\mathbf{P}^{N-2}$, say M' , is smooth with $C' \cong C$ and (under this isomorphism) $\mathcal{O}_{C'}(1) \cong \mathcal{O}_C(1) \otimes \mathcal{O}_C(-P)$. Fix a general line $R \subset M$ with $P \in R$. Assume $N \geq 7$. For general R we may find such a line R with $\{P\} = \text{Sec}(C) \cap R$ (quasi-transversal intersection at P). Fix a general $\mathbf{P}^{N-1}$ and consider the family $\{C_y\}_{y \in R}$ of projections of C into $\mathbf{P}^{N-1}$ from $y \in R$ as y varies. For $y \neq P$, $(C_y, \mathbf{P}^{N-1})$ is projectively equivalent to (C, M) (i.e., under the standard identification of $\mathbf{P}^{N-1}$ and M given by the projection from y , C_y goes to C). For $y = P$ by [Ba-El 1, proof of Proposition I.1], we get that C_P is the union of $C' \subset M'$ and a line E linked to C' (see [Ba-El 1, picture at p. 4]; E is the image of the plane spanned by R and $T_P C$). The condition that S and C are scheme-theoretically cut out by quadrics means exactly that in M there are $N-2$ quadrics of M containing C and whose tangent spaces at P have intersection $T_P C$. This means that the subspace of $H^0(M, \mathcal{I}_{C,M}(2))$ formed by quadrics which are cones of vertex P has codimension $N-2$ (as small as possible). If (as very often in our situation) $h^1(\mathcal{O}_{C'}(2)) = 0$, this means that the restriction map $H^0(M', \mathcal{O}_{M'}(2)) \rightarrow H^0(C', \mathcal{O}_{C'}(2))$ is surjective; very often it is very easy to check the corresponding surjectivity statement for forms of degree > 2 . Using a Mayer-Vietoris exact sequence we get the surjectivity of $H^0(M, \mathcal{O}_M(t)) \rightarrow H^0(C' \cup E, \mathcal{O}_{C' \cup E}(t))$ for every $t \geq 2$.

Thus we find in a suitable W' a complete family (parametrized by $\mathbf{P}^1$). For instance this implies that $W \setminus W'$ does not contain the support of an ample divisor.

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