ON THE RANK AND THE CRANK MODULO 4

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ABSTRACT. Let $N(r, m, n)$ (respectively, $M(r, m, n)$) denote the number of partitions of n whose ranks (respectively, cranks) are congruent to r modulo n . It is shown that $N(0, 4, 2n+1) = M(1, 4, 2n+1)$ and $N(2, 4, 2n) = M(1, 4, 2n)$.

INTRODUCTION

Let

$$\pi = \pi_0 + \pi_1 + \cdots + \pi_{s-1}$$

be a partition with the parts arranged in nonincreasing order. Dyson [D1] defined the rank of π to be the number: $\pi_0 - s$ (the empty partition of 0 has rank 0). Writing $N(r, m, n)$ for the number of partitions of n whose ranks are congruent to r modulo m , he noticed that several relations appeared to hold between the numbers $N(r, m, kn+s)$ when $m = k = 5$ and when $m = k = 7$. Of particular interest are the relations

$$(1) \quad N(0, 5, 5n+4) = N(1, 5, 5n+4) = \cdots = N(4, 5, 5n+4)$$

and

$$(2) \quad N(0, 7, 7n+5) = N(1, 7, 7n+5) = \cdots = N(6, 7, 7n+6),$$

since they flesh out the Ramanujan congruences [R], viz. $p(5n+4) \equiv 0 \pmod{5}$ and $p(7n+5) \equiv 0 \pmod{7}$. All Dyson's observations were shown to be correct by Atkin and Swinnerton-Dyer [ASD].

Ramanujan also showed that $p(11n+6) \equiv 0 \pmod{11}$. However, the values of their ranks mod 11 do not sort the partitions of $11n+6$ into equal piles, and Dyson suggested there might exist a *crank* which would perform this trick. Such a crank was found by Andrews and Garvan [AG]. Here I give the (slightly different) crank of Dyson [D2]. With $t := \pi_0 - \pi_1$, the crank of π is

$$\begin{cases} t - \pi_1, & \text{if } t > 0, \\ -s, & \text{if } t = 0. \end{cases}$$

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Write $M(r, m, n)$ for the number of partitions of n whose cranks are congruent to r modulo m . Then

$$M(0, 11, 11n + 6) = M(1, 11, 11n + 6) = \cdots = M(10, 11, 11n + 6).$$

Also, (1) and (2) hold (with M for N) and there are numerous other relations between the $M(r, m, kn + s)$, when $(m, k) = (5, 5)$, $(7, 7)$ and $(11, 11)$ [G1] and when $(m, k) = (8, 2)$, $(9, 3)$ and $(10, 5)$ [G2]. Furthermore, there are several relations between the M 's and the N 's for $(m, k) = (5, 5)$ and $(7, 7)$ and several conjectural relations for $(m, k) = (8, 4)$, $(9, 3)$ and $(12, 2)$ [L]. I show here

Theorem. For all $n \geq 0$,

$$(3a) \quad N(2, 4, 2n) = M(1, 4, 2n),$$

$$(3b) \quad N(0, 4, 2n + 1) = M(1, 4, 2n + 1).$$

In what follows, q denotes a complex number of modulus less than 1, and z is a nonzero complex number. Also

$$(z)_\infty = (z; q)_\infty := (1 - z)(1 - zq)(1 - zq^2) \cdots, \\ [z; q]_\infty := (z; q)_\infty (z^{-1}q; q)_\infty.$$

Note that

$$(4) \quad [zq; q]_\infty = [z^{-1}; q]_\infty = -z^{-1}[z; q]_\infty.$$

Also,

$$P := (q)_\infty^{-1} = \sum p(n)q^n,$$

where $p(n)$ is the number of partitions of n . Here, and below, $\sum$ denotes a sum over all integers (but note that $p(n) = 0$ for $n < 0$) and $\sum'$ denotes a sum over all nonzero integers. If X is a power series in q , X_{ev} and X_{odd} denote the power series in q^2 with

$$X = X_{\text{ev}} + X_{\text{odd}}q.$$

The techniques I use are closely modelled on those of Atkin and Swinnerton-Dyer [ASD]. They described their methods as old-fashioned, so mine are doubly so.

LEMMAS

Jacobi's triple product identity [A, Theorem 2.8] states that

$$(5) \quad (q)_\infty [z; q]_\infty = \sum (-)^k z^k q^{k(k-1)/2}.$$

Writing q^3 for q and q for z in (5) gives Euler's identity [A, Corollary 1.7]:

$$(6) \quad P^{-1} = (q)_\infty = \sum (-)^k q^{k(3k-1)/2}.$$

Another identity I shall use is the quintuple product identity [Go]:

$$(7) \quad (q)_{\infty}[z; q]_{\infty}[z^2q; q^2]_{\infty} = (q^3; q^3)_{\infty}([-z^3q; q^3]_{\infty} - z[-z^3q^2; q^3]_{\infty}).$$

Sorting the right-hand side of (6) into even and odd powers of q and appealing to (5) and (7) gives

$$(8) \quad (q)_{\infty} = (q^{16}; q^{16})_{\infty}(q^2; q^4)_{\infty}([-q^6; q^{16}]_{\infty} - [-q^2; q^{16}]_{\infty}q).$$

Multiplying (8) by the result of replacing q by $-q$ in (8) shows that

$$(9) \quad [-q^6; q^{16}]_{\infty}^2 - [-q^2; q^{16}]_{\infty}^2 q^2 = (q^2; q^4)_{\infty}[q^4; q^8]_{\infty}[q^8; q^{16}]_{\infty}$$

and from (8) and (9) we have

$$(10) \quad P = \frac{(q^{16}; q^{16})_{\infty}}{(q^2; q^2)_{\infty}^2}([-q^6; q^{16}]_{\infty} + [-q^2; q^{16}]_{\infty}q).$$

Lemma 1.

$$(11) \quad [-q^2; q^4]_{\infty}^2 = [q; q^2]_{\infty}^2[q^2; q^4]_{\infty} + [-q^4; q^4]_{\infty}^2 q.$$

Proof. From (5), we have

$$(12) \quad [-q; q^2]_{\infty} = (-q^4; q^8)_{\infty} + [-q^8; q^8]_{\infty}q(q^2; q^4)_{\infty}^{-1}(q^4; q^8)_{\infty}^{-1}.$$

Set

$$f(q) = [-q^2; q^4]_{\infty}^2 - [q; q^2]_{\infty}^2[q^2; q^4]_{\infty} - [-q^4; q^4]_{\infty}^2 q.$$

From (12),

$$\begin{aligned} f(-q) &= [-q^2; q^4]_{\infty}^2 + [-q^4; q^4]_{\infty}^2 q \\ &\quad - ([-q^4; q^8]_{\infty} + [-q^8; q^8]_{\infty}q)^2 [q^4; q^8]_{\infty}^{-1} \\ &= [q^4; q^8]_{\infty}^{-1}([-q^2; q^4]_{\infty}^2 [q^4; q^8]_{\infty} - [-q^4; q^8]_{\infty}^2 - [-q^8; q^8]_{\infty}^2 q^2) \\ &= -[q^4; q^8]_{\infty}^{-1} f(-q^2). \end{aligned}$$

Since $f(-q)$ has zero constant term, it is identically zero. So $f(q)$ is zero and (11) follows. $\square$

Now define (for any complex ζ)

$$T(z, \zeta, q) := \sum (-)^n \frac{\zeta^n q^{3n(n+1)/2}}{(1 - zq^n)}.$$

$T(z, \zeta, q)$ is an analytic function of z in every region $0 < z_1 \leq z \leq z_2$, except for simple poles where $z = q^n$. Also, let

$$T_1(\zeta, q) := \sum' (-)^n \frac{\zeta^n q^{3n(n+1)/2}}{(1 - q^n)},$$

and let

$$g(z) = g(z, q) := z \frac{[z^2; q]_{\infty}}{[z; q]_{\infty}} T(z, 1, q) - z^3 T(z^2, z^3, q) - T_1(z^{-3}, q).$$

Atkin and Swinnerton-Dyer [ASD, (5.1)] showed that

$$(13) \quad \zeta^3 T(z\zeta, \zeta^3, q) + T(z\zeta^{-1}, \zeta^{-3}, q) = \zeta \frac{[\zeta^2; q]_\infty}{[\zeta; q]_\infty} T(z, 1, q) \\ + \frac{[\zeta; q]_\infty [\zeta^2; q]_\infty (q; q)_\infty^2}{[z\zeta; q]_\infty [z; q]_\infty [z\zeta^{-1}; q]_\infty}$$

that [ASD, (5.11)]

$$(14) \quad g(z) - g(zq) = -3$$

and that [ASD, (5.7)]

$$(15) \quad g(z) + g(z^{-1}q) = 1.$$

They also gave an expression for $2g(z) - g(z^2)$ in terms of theta functions, but I need

Lemma 2.

$$(16) \quad 3g(z) - g(z^3) = \frac{[z^4; q]_\infty^3 (q; q)_\infty^2}{[z^2; q]_\infty^3 [z^6; q]_\infty} + \frac{[z^2; q]_\infty^3 (q; q)_\infty^2}{[z; q]_\infty^3 [z^3; q]_\infty} - 2.$$

Proof.

Let

$$H(z) := \frac{[z^2; q]_\infty^3 (q; q)_\infty^2}{[z; q]_\infty^3 [z^3; q]_\infty}.$$

By (14), $3g(z) - g(z^3)$ is unchanged under $z \mapsto zq$, as are $H(z)$ and $H(z^2)$ (by (4)). $H(z)$ and $H(z^2)$ together eliminate the poles of $3g(z) - g(z^3)$ and so

$$(17) \quad 3g(z) - g(z^3) - H(z) - H(z^2) = c,$$

where c is a constant [ASD, Lemma 2]. To determine c , note that, from (14) and (15),

$$(18) \quad g(z) + g(z^{-1}) = -2.$$

Since

$$H(-1) = 0 \quad \text{and} \quad H(-i) = -H(i),$$

adding the results of setting $z = i$ and $z = -i$ in (17) and using (18) gives $c = -2$. $\square$

THE PROOF

Let $N(m, n)$ (respectively $M(m, n)$) denote the number of partitions of n with rank (respectively crank) m . Then [D1, (20)]

$$(19) \quad \sum \sum N(m, n) z^m q^n = \frac{(1-z)}{(q)_\infty} \sum \frac{(-)^n q^{n(3n+1)/2}}{(1-zq^n)}.$$

(Note that, since we suppose the empty partition to have rank 0, $N(0, 0) = 1$ in contrast to Dyson [D1] and Atkin and Swinnerton-Dyer [ASD], for whom $N(0, 0) = 0$.)

For integer a , let

$$S(a) := \sum' \frac{(-)^n q^{n(3n+1)/2+an}}{(1-q^{4n})}.$$

Setting $z = i$ in (19), we find that

$$(20a) \quad A := \sum (N(0, 4, n) - N(2, 4, n))q^n = P(2S(0) + 2S(1) + 1)$$

(note that $N(r, m, n) = N(m-r, m, n)$), and setting $z = -1$ in (19) gives

$$(20b) \quad B := \sum (N(0, 2, n) - N(1, 2, n))q^n = P(4S(0) - 4S(1) + 1).$$

Also [AG],

$$(21) \quad \sum \sum M(m, n) z^m q^n = q - z^{-1}q + \frac{(q)_\infty}{(zq)_\infty (z^{-1}q)_\infty}$$

and so (setting $z = -1$ in (21))

$$(22) \quad \sum (M(0, 2, n) - M(1, 2, n))q^n = 2q + C,$$

where

$$(20c) \quad C := \frac{(q)_\infty}{(-q)_\infty^2}.$$

Since $P = \sum (N(0, 2, n) + N(1, 2, n))q^n$,

$$(23a) \quad P + B - 2A = 4 \sum N(2, 4, n)q^n,$$

$$(23b) \quad P + B + 2A = 4 \sum N(0, 4, n)q^n.$$

Also, $P = \sum (M(0, 2, n) + M(1, 2, n))q^n$ and (22) gives

$$P - 2q - C = 2 \sum M(1, 2, n)q^n = -2q + 4 \sum M(1, 4, n)q^n$$

(since $M(1, 2, n) = M(1, 4, n) + M(3, 4, n)$, and, for $n > 1$, $M(1, 4, n) = M(3, 4, n)$) and so

$$(23c) \quad P - C = 4 \sum M(1, 4, n)q^n.$$

It follows from (23a)–(23c) that the theorem will be proved once we show that

$$(24a) \quad (2A - B)_{\text{ev}} = C_{\text{ev}}$$

and

$$(24b) \quad (2A + B)_{\text{odd}} = -C_{\text{odd}}.$$

First we find expressions for C_{ev} and C_{odd} . From (8) and (20c) we have

$$C = \frac{(q)_\infty^3}{(q^2; q^2)_\infty^2} = \frac{(q^{16}; q^{16})_\infty^3 (q^2; q^4)_\infty^3}{(q^2; q^2)_\infty^2} ([-q^6; q^{16}]_\infty - [-q^2; q^{16}]_\infty q)^3$$

and so

$$(25a) \quad C_{\text{ev}} = \frac{(q^{16}; q^{16})_{\infty}^3 (q^2; q^4)_{\infty}^3}{(q^2; q^2)_{\infty}^2} ([-q^6; q^{16}]_{\infty}^3 + 3[-q^6; q^{16}]_{\infty} [-q^2; q^{16}]_{\infty}^2 q^2)$$

and

$$(25b) \quad C_{\text{odd}} = -\frac{(q^{16}; q^{16})_{\infty}^3 (q^2; q^4)_{\infty}^3}{(q^2; q^2)_{\infty}^2} (3[-q^6; q^{16}]_{\infty}^2 [-q^2; q^{16}]_{\infty} + [-q^2; q^{16}]_{\infty}^3 q^2).$$

Next we find expressions for $(2A - B)_{\text{ev}}$ and $(2A + B)_{\text{odd}}$. (20a) and (20b) give

$$(26a) \quad 2A - B = P(8S(1) + 1)$$

and

$$(26b) \quad 2A + B = P(8S(0) + 3) = 8 + P(8S(4) - 5)$$

(because $S(0) - S(4) = \sum' (-)^k q^{k(3k-1)/2} = P^{-1} - 1$ (see (6)).

Straightforward manipulation gives

$$(27a) \quad S(1) = T_1(-q^{-18}, q^{16}) - q^{18} T(q^{12}, -q^{18}, q^{16}) \\ + (q^6 T(q^8, -q^6, q^{16}) - T(q^4, -q^{-6}, q^{16})) q^3$$

and

$$(27b) \quad S(4) = T_1(-q^{-6}, q^{16}) - q^6 T(q^4, -q^6, q^{16}) \\ + (q^{18} T(q^8, -q^{18}, q^{16}) - T(q^{-4}, -q^{-18}, q^{16})) q^{-3}.$$

Now (10), (13), (26a), (26b), (27a), and (27b) thrown together give

$$(28a) \quad (2A - B)_{\text{ev}} = \frac{(q^{16}; q^{16})_{\infty}}{(q^2; q^2)_{\infty}^2} \left([-q^6; q^{16}]_{\infty} (1 - 8g(-q^6)) \right. \\ \left. - 8 \frac{[-q^2; q^{16}]_{\infty}^2 (q^{16}; q^{16})_{\infty}^2}{[-q^6; q^{16}]_{\infty} [q^8; q^{16}]_{\infty}} q^4 \right)$$

and

$$(28b) \quad (2A + B)_{\text{odd}} = \frac{(q^{16}; q^{16})_{\infty}}{(q^2; q^2)_{\infty}^2} \left([-q^2; q^{16}]_{\infty} (-5 - 8g(-q^2)) \right. \\ \left. + 8 \frac{[-q^6; q^{16}]_{\infty}^2 (q^{16}; q^{16})_{\infty}^2}{[-q^2; q^{16}]_{\infty} [q^8; q^{16}]_{\infty}} \right).$$

From (15) (with $z = -q^6$ and with q^{16} for q) we have

$$(29) \quad 3g(-q^6) - g(-q^{18}) \\ = \left(-\frac{[q^8; q^{16}]_{\infty}^3}{[q^4; q^{16}]_{\infty}^4} + \frac{[q^4; q^{16}]_{\infty}^3}{[-q^6; q^{16}]_{\infty}^3 [-q^2; q^{16}]_{\infty}} q^2 \right) (q^{16}; q^{16})_{\infty}^2 - 2$$

and (with $z = -q^2$)

$$(30) \quad \begin{aligned} & 3g(-q^2) - g(-q^6) \\ &= \left(\frac{[q^8; q^{16}]_\infty^3}{[q^4; q^{16}]_\infty^4} + \frac{[q^4; q^{16}]_\infty^3}{[-q^2; q^{16}]_\infty^3 [-q^6; q^{16}]_\infty} \right) (q^{16}; q^{16})_\infty^2 - 2. \end{aligned}$$

From (14), (29), and (30) we find that

$$(31a) \quad \begin{aligned} 1 - 8g(-q^6) &= \left(2 \frac{[q^8; q^{16}]_\infty^3}{[q^4; q^{16}]_\infty^4} - \frac{[q^4; q^{16}]_\infty^3}{(-q^2; q^4)_\infty^3} \right. \\ &\quad \times ([-q^6; q^{16}]_\infty^2 + 3[-q^2; q^{16}]_\infty^2 q^2) \left. \right) (q^{16}; q^{16})_\infty^2 \end{aligned}$$

and

$$(31b) \quad \begin{aligned} -5 - 8g(-q^2) &= \left(-2 \frac{[q^8; q^{16}]_\infty^3}{[q^4; q^{16}]_\infty^4} - \frac{[q^4; q^{16}]_\infty^3}{(-q^2; q^4)_\infty^3} \right. \\ &\quad \times (3[-q^6; q^{16}]_\infty^2 + [-q^2; q^{16}]_\infty^2 q^2) \left. \right) (q^{16}; q^{16})_\infty^2 \end{aligned}$$

Inserting (31a), (31b) into (28a), (28b) (note that $[q^8; q^{16}]_\infty [q^4; q^{16}]_\infty^{-2} = [-q^4; q^8]_\infty$) gives

$$(32a) \quad \begin{aligned} (2A - B)_{\text{ev}} &= \frac{(q^{16}; q^{16})_\infty^3}{(q^2; q^2)_\infty^2} [-q^6; q^{16}]_\infty \\ &\quad \times \left(2[q^8; q^{16}]_\infty [-q^4; q^8]_\infty^2 - ([-q^6; q^{16}]_\infty^2 \right. \\ &\quad \left. + 3[-q^2; q^{16}]_\infty^2 q^2)(q^2; q^4)_\infty^3 - 8 \frac{[-q^2; q^{16}]_\infty^2}{[-q^6; q^{16}]_\infty^2 [q^8; q^{16}]_\infty} q^4 \right) \end{aligned}$$

and

$$(32b) \quad \begin{aligned} (2A + B)_{\text{odd}} &= \frac{(q^{16}; q^{16})_\infty^3}{(q^2; q^2)_\infty^2} [-q^2; q^{16}]_\infty \\ &\quad \times \left(-2[q^8; q^{16}]_\infty [-q^4; q^8]_\infty^2 - (3[-q^6; q^{16}]_\infty^2 \right. \\ &\quad \left. + [-q^2; q^{16}]_\infty^2 q^2)(q^2; q^4)_\infty^3 + 8 \frac{[-q^6; q^{16}]_\infty^2}{[-q^2; q^{16}]_\infty^2 [q^8; q^{16}]_\infty} \right). \end{aligned}$$

If we now compare (32a), (32b) with (25a), (25b), we see that (24a) (and so also (3a)) holds if

$$\begin{aligned} X &:= [q^8; q^{16}]_\infty [-q^4; q^8]_\infty^2 - ([-q^6; q^{16}]_\infty^2 + 3[-q^2; q^{16}]_\infty^2 q^2)(q^2; q^4)_\infty^3 \\ &\quad - 4 \frac{[-q^2; q^{16}]_\infty^2}{[-q^6; q^{16}]_\infty^2 [q^8; q^{16}]_\infty} q^4 = 0 \end{aligned}$$

and that (24b) and (3b) hold if

$$Y := [q^8; q^{16}]_{\infty} [-q^4; q^8]_{\infty}^2 + (3[-q^6; q^{16}]_{\infty}^2 + [-q^2; q^{16}]_{\infty}^2 q^2)(q^2; q^4)_{\infty}^3 \\ - 4 \frac{[-q^6; q^{16}]_{\infty}^2}{[-q^2; q^{16}]_{\infty}^2 [q^8; q^{16}]_{\infty}} = 0.$$

But

$$Y - X = 4([-q^6; q^{16}]_{\infty}^2 + [-q^2; q^{16}]_{\infty}^2 q^2) \\ \times \left((q^2; q^4)_{\infty}^3 - \frac{1}{[-q^2; q^4]_{\infty} [q^8; q^{16}]_{\infty}} ([-q^6; q^{16}]_{\infty}^2 - [-q^2; q^{16}]_{\infty}^2 q^2) \right) \\ = 0$$

by (9), and

$$Y + X = 2 \left([q^8; q^{16}]_{\infty} [-q^4; q^8]_{\infty}^2 + ([-q^6; q^{16}]_{\infty}^2 - [-q^2; q^{16}]_{\infty}^2 q^2)(q^2; q^4)_{\infty}^3 \right. \\ \left. - \frac{2}{[-q^2; q^4]_{\infty} [q^8; q^{16}]_{\infty}} ([-q^6; q^{16}]_{\infty}^4 + [-q^2; q^{16}]_{\infty}^4 q^4) \right) \\ = 2[q^8; q^{16}]_{\infty} ([-q^4; q^8]_{\infty}^2 - [q^2; q^4]_{\infty}^2 [q^4; q^8]_{\infty} - [-q^8; q^8]_{\infty}^2 q^2)$$

(using (9))

$$= 0,$$

by (11). So $X = Y = 0$ and the theorem is proved. $\square$

SOME QUESTIONS

In the following list, the “=” have been established by Garvan [G2] and the “=” follow from the theorem proved above. The “ $\stackrel{?}{=}$ ” are guesses that I cannot prove [L].

$$M(1, 8, 4n) = M(3, 8, 4n) =^* N(2, 8, 4n) \stackrel{?}{=} N(4, 8, 4n),$$

$$M(2, 8, 4n) \stackrel{?}{=} N(3, 8, 4n).$$

$$M(0, 8, 4n+1) + M(1, 8, 4n+1) = M(3, 8, 4n+1) + M(4, 8, 4n+1) \\ \stackrel{?}{=} N(1, 8, 4n+1) + N(2, 8, 4n+1) \\ \stackrel{?}{=} N(3, 8, 4n+1) + N(4, 8, 4n+1),$$

$$M(2, 8, 4n+1) \stackrel{?}{=} N(3, 8, 4n+1).$$

$$M(1, 8, 4n+2) = M(3, 8, 4n+2) =^* N(2, 8, 4n+2) \stackrel{?}{=} N(0, 8, 4n+2),$$

$$M(2, 8, 4n+2) \stackrel{?}{=} N(1, 8, 4n+2).$$

$$\begin{aligned}
M(0, 8, 4n+3) + M(1, 8, 4n+3) &= M(3, 8, 4n+3) + M(4, 8, 4n+3) \\
&\stackrel{?}{=} N(0, 8, 4n+3) + N(1, 8, 4n+3) \\
&\stackrel{?}{=} N(2, 8, 4n+3) + N(3, 8, 4n+3), \\
M(2, 8, 4n+3) &\stackrel{?}{=} N(1, 8, 4n+3).
\end{aligned}$$

Using the techniques of this paper, I can reduce the “ $\stackrel{?}{=}$ ” to certain “identities” between theta functions. I have considerable computer evidence for these “identities” but, as yet, I am unable to nail them down.

Note. Nicolas Santa-Gadea (thesis, Pennsylvania State University) has found proofs of all the “ $\stackrel{?}{=}$ ” of the last section. He uses the third order mock theta functions. I also have proofs using the classical theory of modular forms.

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