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ON r-SEPARATED SETS IN NORMED SPACES

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ABSTRACT. The *separation* of a bounded set A in a metric space $\delta(A)$ is defined as the supremum of the numbers $r > 0$ such that there exists a sequence (x_n) in A such that $d(x_n, x_m) > r$ for every $n \neq m$. We prove for every bounded set A in a Banach space that $\delta(A) = \delta(\text{co}(A))$ where $\text{co}(A)$ denotes the convex hull of A. This yields a generalization of Darbo's fixed point theorem.

1. INTRODUCTION

In 1939 Kuratowski [10] introduced the measure of noncompactness $\alpha(A)$ of a bounded set A in a metric space X. $\alpha(A)$ is called the *Kuratowski measure of noncompactness*, and is defined as the greatest lower bound of the numbers $r > 0$ such that A can be decomposed into a finite union of *sets* of diameter smaller than r. The condition $\alpha(A) = 0$ therefore means that A is precompact. Another measure of noncompactness, which in many cases seems to be more convenient, is called the *ball-measure*, $\beta(A)$, and is defined as the infimum of the real numbers $r > 0$ such that there is a finite cover of A with *balls* of radii smaller than r.

These and other measures of noncompactness were used by Darbo [2], Massat [11], Sadovskiĭ [12], and Banaś and Goebel [1] to obtain some fixed point theorems of nonlinear maps. In general, *a measure of noncompactness* on a complete metric space X is a function γ which maps every bounded set $B \subset X$ to a positive real number $\gamma(B)$ such that:

(a) $\gamma(B) = 0$ if and only if $\overline{B}$ is compact;
(b) if $B \subset C$ are bounded sets, then $\gamma(B) \leq \gamma(C)$.

Furthermore, if B is a closed convex bounded subset of a Banach space X, an operator $T: B \to B$ is called γ*-condensing* if for all bounded sets $C \subset B$ we have $\gamma(T(C)) \leq \gamma(C)$ with equality if and only if $\gamma(C) = 0$.

The following theorem (Sadovskiĭ [12], Massat [11]) illustrates the utility of these concepts.

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Theorem. *Let X be a Banach space and $B \subset X$ a closed convex bounded set. Let $T: B \to B$ be a continuous and γ-condensing operator, where γ is a measure of noncompactness on X such that*

(c) $\gamma(A \cup B) = \gamma(A)$ *for every finite set* $B \subset X$;
(d) $\gamma(A) = \gamma(\mathrm{co}(A))$ *for every bounded set* $A \subset X$, *where* $\mathrm{co}(A)$ *denotes the convex hull of* A.

Then T has a fixed point.

In Wells and Williams [13], another measure of noncompactness is defined: the *separation* of A, $\delta(A)$, is the supremum of the numbers $r > 0$ such that there exists a sequence (x_n) in A such that $d(x_n, x_m) > r$ for every $n \neq m$. They use it because this measure can distinguish between the unit balls of Banach spaces. The measure of the unit ball seems to be connected with the reflexivity of a Banach space (Kottman [9]). And the theorem of Elton and Odell [6] insures that $\delta(B) > 1$ for the unit ball B of every infinite dimensional Banach space.

This same concept of separation of A has been defined independently by Domínguez Benavides [3] and is denoted by $\mu(A)$. He observes [4] that in some Banach spaces as the l_p spaces $(1 < p < +\infty)$, $\delta(A) = \mu(A)$ is proportional to the β-measure. In other spaces, say $L^p[0, 1]$ $(p \neq 2)$, this relation is not satisfied.

Observe that the above fixed point theorem would apply to this measure of noncompactness if we proved that

$$\delta(A) = \delta(\mathrm{co}(A))$$

for every bounded subset A of a Banach space. The purpose of this paper is to prove this. It is important to point out that, in a recent paper, Domínguez Benavides [5] proves that every α-contraction is a δ-contraction, so the fixed point theorem obtained in our paper generalizes the fixed point theorem of Darbo [2].

In §2 we give a probabilistic lemma (Corollary 2) that plays an essential role in the proof of the main theorem (Theorem 5). In that proof we also use an easier version of a deep theorem of D. H. Fremlin and M. Talagrand about random graphs, which we enunciate as Theorem 3.

2. Probabilistic lemmas

Lemma 1. *Let μ, ν be two probability measures on the space $[-1, 1]$ such that*

$$\int_{-1}^{1} x\, d\mu(x) - \int_{-1}^{1} x\, d\nu(x) \geq s > \theta > 1,$$

then there exists a real number $t_0 \in [\theta - 1, 1]$ such that

$$\mu[-1, t_0) + \nu[t_0 - \theta, 1] \leq (2 - s)/(2 - \theta).$$

Proof. Observe that an elementary calculation leads us to

$$\int_{-1}^{1}(1+x)\,d\mu(x)=\int_{0}^{2}\mu[-1+t\,,\,1]\,dt=\int_{-1}^{1}\mu[t\,,\,1]\,dt.$$

It follows that

$$\int_{-1}^{1}x\,d\mu(x)=1-\int_{-1}^{1}\mu[-1\,,\,t)\,dt$$

$$\int_{-1}^{1}x\,d\nu(x)=\int_{-1}^{1}\nu[t\,,\,1]\,dt-1.$$

Therefore

$$\begin{aligned}\int_{-1}^{1}x\,d\mu(x)-\int_{-1}^{1}x\,d\nu(x)&=2-\int_{-1}^{1}\mu[-1\,,\,t)-\int_{-1}^{1}\nu[t\,,\,1]\,dt\\&=2-\int_{-1}^{1}\mu[-1\,,\,t)-\int_{-1+\theta}^{1+\theta}\nu[t-\theta\,,\,1]\,dt\\&\le 2-\int_{-1+\theta}^{1}(\mu[-1\,,\,t)+\nu[t-\theta\,,\,1])\,dt.\end{aligned}$$

Define $\psi(t)=(\mu[-1\,,\,t)+\nu[t-\theta\,,\,1])$. By the hypothesis we know that

$$2-s\ge\int_{-1+\theta}^{1}\psi(t)\,dt.$$

Then $\psi(t)>(2-s)/(2-\theta)$ for every $t\in[-1+\theta\,,\,1]$ would lead to a contradiction. It follows that there exists $t_0\in[-1+\theta\,,\,1]$ such that $\psi(t_0)\le(2-s)/(2-\theta)$.

We shall use the following corollary in the proof of the main theorem.

Corollary 2. *Let* $(\Omega_1\,,\,\mathbb{P}_1)$, $(\Omega_2\,,\,\mathbb{P}_2)$ *be two probability spaces and* $X_i\colon\Omega_i\to E$ *two random variables with values in the same Banach space* E. *Assume that* $\|X_i\|_\infty\le 1$ *and*

$$\|\mathbb{E}(X_1)-\mathbb{E}(X_2)\|\ge s>\theta>1.$$

Then there exist two measurable subsets $A\subset\Omega_1$ *and* $B\subset\Omega_2$ *such that*

(1) $$\mathbb{P}_1(A)+\mathbb{P}_2(B)\le(2-s)/(2-\theta)<1.$$

(2) $$\omega_1\notin A \text{ and } \omega_2\notin B \text{ implies } \|X_1(\omega_1)-X_2(\omega_2)\|>\theta.$$

Proof. There is no loss of generality in assuming that the Banach space E is real. Let $x^*\in E^*$ be a vector of the dual space such that $\|x^*\|=1$ and

$$\|\mathbb{E}(X_1)-\mathbb{E}(X_2)\|=\mathbb{E}(x^*\circ X_1)-\mathbb{E}(x^*\circ X_2).$$

Let μ and ν be the image of the measures $\mathbb{P}_1$ and $\mathbb{P}_2$ under the mappings $x^*\circ X_1$ and $x^*\circ X_2$. Since $\|X_i\|_\infty\le 1$, μ and ν are probability measures on $[-1\,,\,1]$. Observe that

$$\int_{-1}^{1}x\,d\mu(x)-\int_{-1}^{1}x\,d\nu(x)=\mathbb{E}(x^*\circ X_1)-\mathbb{E}(x^*\circ X_2)\ge s>\theta>1.$$

Applying the lemma we find $t_0 \in [\theta - 1, 1]$ such that

$$\mu[-1, t_0) + \nu[t_0 - \theta, 1] \leq (2-s)/(2-\theta) < 1.$$

Define $A = (x^* \circ X_1)^{-1}[-1, t_0)$ and $B = (x^* \circ X_2)^{-1}[t_0 - \theta, 1]$. Thus $A \subset \Omega_1$ and $B \subset \Omega_2$ are measurable sets that satisfy (1).

If $\omega_1 \notin A$ and $\omega_2 \notin B$, then $x^*(X_1(\omega_1)) \geq t_0$ and $x^*(X_2(\omega_2)) < t_0 - \theta$. It follows that

$$\|X_1(\omega_1) - X_2(\omega_2)\| \geq x^*(X_1(\omega_1) - X_2(\omega_2)) > \theta$$

and (2) is also satisfied.

We shall use the following consequence of a theorem of D. H. Fremlin and M. Talagrand [7]:

Theorem 3. *Let $(\Omega_n, \mathbb{P}_n)$ be a sequence of probability spaces. For every $m < n$, let $B_{n,m} \subset \Omega_n$ be a measurable set such that $\mathbb{P}_n(B_{n,m}) \leq \alpha < 1$. Then there exists an infinite set $J \subset \mathbf{N}$ such that, for every $n \in J$,*

$$\mathbb{P}_n\left(\bigcup_{\substack{m<n \\ m\in J}} B_{n,m}\right) < 1.$$

Proof. We apply the theorems 6C and 6D [7] to the probability space $\Omega = \prod_{j=1}^{\infty} \Omega_j$ and the measurable sets

$$E_{n,m} = \Omega \setminus \left(B_{n,m} \times \prod_{j\neq n} \Omega_j\right)$$

in order to obtain the theorem.

3. Theorem about the measure of noncompactness

Let us recall that if $A \subset X$ is a bounded subset in a metric space X, the separation of A, $\delta(A)$ is defined as the supremum of the real numbers $r > 0$ such that there exists a sequence (x_n) in A verifying $\|x_n - x_m\| > r$ for every two distinct elements x_n, x_m of the sequence.

We begin making a reduction in the possible counterexample to $\delta(A) = \delta(\text{co}(A))$.

Proposition 4. *Let A be a bounded subset of the normed space X such that $\delta(A) < \delta(\text{co}(A))$. Then, for every s verifying $\delta(A) < s < \delta(\text{co}(A))$, there exists a set B contained in the ball of center 0 and radius s and such that*

$$\delta(B) \leq \delta(A) < \delta(\text{co}(A)) = \delta(\text{co}(B)).$$

Proof. Take a maximal set $\{a_k\}_{k=1}^{N}$ of points belonging to A such that $i \neq j$ implies $\|a_i - a_j\| \geq s$. The number N of elements is finite since $\delta(A) < s$.

It is clear that A is contained in the union of the balls $B(a_k, s)$ of center a_k and radius s:

$$A \subset \bigcup_{k=1}^{N} B(a_k, s).$$

Every point $x \in \mathrm{co}(A)$ can be written in the form $x = \sum_{k=1}^{N} \alpha_k x_k$, where $\alpha_k \geq 0$, $\sum \alpha_k = 1$ and $x_k \in \mathrm{co}(A \cap B(a_k, s))$. Now, given $\varepsilon > 0$, let $\{z_n\}_{n=1}^{\infty}$ be a sequence of points in $\mathrm{co}(A)$ such that $n \neq m$ implies $\|z_n - z_m\| > \delta(\mathrm{co}(A)) - \varepsilon$.

Write every z_n in the form:

$$z_n = \sum_{k=1}^{N} \alpha_k^n y_k^n,$$

where $\alpha_n^k \geq 0$, $\sum_{k=1}^{N} \alpha_k^n = 1$, and $y_k^n \in \mathrm{co}(A \cap B(a_k, s))$. Choose an infinite subset $I \subset \mathbf{N}$ such that if $n, m \in I$, then for all $k \leq N$,

$$|\alpha_k^n - \alpha_k^m| < \varepsilon/(N \cdot \sup_{a \in A} \|a\|).$$

Now let $n, m \in I$ and $n \neq m$. Then there exists a natural number k such that $\|y_k^n - y_k^m\| > \delta(\mathrm{co}(A)) - 2\varepsilon$, because otherwise

$$\begin{aligned} \|z_n - z_m\| &\leq \left\| \sum_{k=1}^{N} \alpha_k^n (y_k^n - y_k^m) + (\alpha_k^n - \alpha_k^m) y_k^m \right\| \\ &\leq (\delta(\mathrm{co}(A)) - 2\varepsilon) \cdot 1 + \sup_{a \in A} \|a\| \sum_{k=1}^{N} |\alpha_k^n - \alpha_k^m| \\ &< \delta(\mathrm{co}(A)) - \varepsilon, \end{aligned}$$

which contradicts the hypothesis.

Applying the Ramsey theorem, T. Jech [8], we obtain an infinite set J contained in I and an index k such that $n, m \in J$, $n \neq m$ imply $\|y_k^n - y_k^m\| \geq \delta(\mathrm{co}(A)) - 2\varepsilon$. Therefore, for every $\varepsilon > 0$, there exists an index k, $1 \leq k \leq N$, such that $\delta(\mathrm{co}(A \cap B(a_k, s))) \geq \delta(\mathrm{co}(A)) - 2\varepsilon$. Hence there exists an index k such that

$$\delta(\mathrm{co}(A \cap B(a_k, s))) \geq \delta(\mathrm{co}(A)).$$

Furthermore we observe that $\delta(A \cap B(a_k, s)) \leq \delta(A)$. We can translate the set $A \cap B(a_k, s)$ and obtain the set $B = -a_k + A \cap B(a_k, s)$, which is contained in the ball of center 0 and radius s and satisfies

$$\delta(B) \leq \delta(A) < \delta(\mathrm{co}(A)) \leq \delta(\mathrm{co}(B)).$$

Theorem 5. *Let $A \subset X$ be a bounded subset of a normed space X. Then $\delta(A) = \delta(\mathrm{co}(A))$.*

Proof. Suppose that the theorem is false. Then Proposition 4 implies that there exists a subset A of the unit ball of a Banach space E, such that

$$\delta(A) < 1 < s < \delta(\mathrm{co}(A)).$$

We can take a sequence (x_n) in $\mathrm{co}(A)$ such that for every $n \neq m$, $\|x_n - x_m\| \geq s$. As $x_n \in \mathrm{co}(A)$, there exists a finite subset $\Omega_n \subset A$ and, for every $e \in \Omega_n$, a real number $\alpha_e > 0$ such that

$$x_n = \sum_{e\in\Omega_n} \alpha_e e \quad \text{and} \quad \sum_{e\in\Omega_n} \alpha_e = 1.$$

Let $\mathbb{P}_n$ be the probability defined on Ω_n by

$$\mathbb{P}_n(B) = \sum_{e\in B} \alpha_e,$$

for every $B \subset \Omega_n$.

For every n, let $X_n : \Omega_n \to E$ be the random variable, defined as the identity in Ω_n. It is clear that if we choose θ such that $1 < \theta < s$, we have, for every $n \neq m$,

$$\|\mathbb{E}(X_n) - \mathbb{E}(X_m)\| = \|x_n - x_m\| > s > \theta > 1.$$

We are now in a position to apply our probabilistic lemma and find subsets $B_{n,m} \subset \Omega_n$ and $B_{m,n} \subset \Omega_m$ verifying:

(a) $\mathbb{P}_n(B_{n,m}) + \mathbb{P}_m(B_{m,n}) \leq (2-s)/(2-\theta) < 1$.

(b) $e_n \notin B_{n,m}$ and $e_m \notin B_{m,n}$ implies $\|e_n - e_m\| \geq \theta$.

For every natural number m_0, $(B_{m_0,n})_{n>m_0}$ is a sequence of subsets of the finite set Ω_{m_0}. It follows that we can find an infinite set $J \subset \mathbf{N}$ such that $B_{m_0,n}$ is independent of $n \in J$. By a diagonal argument we can obtain an infinite set $J \subset \mathbf{N}$ such that if $n > m$, $n, m \in J$, $B_{m,n}$ is independent of n. We call it B_m. Now by exchanging the sequence $(x_n)_{n\in\mathbf{N}}$ for $(x_n)_{n\in J}$, we can also assume that the original sequence satisfies these conditions.

Let $\delta = (2-s)/(2-\theta) < 1$, and choose $0 < \varepsilon < (1-\delta)/2$. We can assume that $\lim \mathbb{P}_n(B_n)$ exists, call it l. Thus $l = \lim \mathbb{P}_n(B_n) \leq 1$; even more, we can assume $|\mathbb{P}_n(B_n) - l| < \varepsilon$ for every natural number n.

Now, for every pair of natural numbers $n > m$

$$\mathbb{P}_n(B_{n,m}) + \mathbb{P}_m(B_m) \leq (2-s)/(2-\theta) = \delta < 1.$$

Hence $l \leq \delta$ and $l - \varepsilon < \mathbb{P}_m(B_m) \leq \delta$.

Therefore we can consider the spaces $\Omega'_n = \Omega_n \setminus B_n$ endowed with the measures $\mathbb{P}'_n = (1 - \mathbb{P}_n(B_n))^{-1}\mathbb{P}_n$. This measure is well defined because $\mathbb{P}_n(B_n) \leq \delta < 1$. Now, for every $m < n$,

$$\begin{aligned}
\mathbb{P}'_n(B_{n,m} \setminus B_n) &= \frac{\mathbb{P}_n(B_{n,m} \setminus B_n)}{1 - \mathbb{P}_n(B_n)} \leq \frac{\mathbb{P}_n(B_{n,m})}{1 - \mathbb{P}_n(B_n)} \leq \frac{\delta - \mathbb{P}_m(B_m)}{1 - \mathbb{P}_n(B_n)} \\
&\leq \frac{1 - \mathbb{P}_n(B_n) + (\mathbb{P}_n(B_n) - \mathbb{P}_m(B_m)) - (1-\delta)}{1 - \mathbb{P}_n(B_n)} \\
&\leq 1 - \frac{1 - \delta - 2\varepsilon}{1 - \mathbb{P}_n(B_n)} \leq 1 - \frac{1 - \delta - 2\varepsilon}{1 - l + \varepsilon} \\
&= 1 - \alpha < 1.
\end{aligned}$$

Hence the conditions in the hypothesis of the D. H. Fremlin and M. Talagrand theorem are satisfied. So, we obtain an infinite set $J \subset \mathbf{N}$ such that, for every $n \in J$,

$$\mathbb{P}'_n \left(\bigcup_{\substack{m<n \\ m\in J}} (B_{n,m} \setminus B_n) \right) < 1 .$$

Now, if we put $B_n = B_{n,n}$, it follows that

$$\mathbb{P}_n \left(\bigcup_{\substack{m\leq n \\ m\in J}} B_{n,m} \right) < 1 .$$

Thus there exists, for every $n \in J$, $e_n \in \Omega_n$ such that

$$e_n \notin \bigcup_{\substack{m\leq n \\ m\in J}} B_{n,m} .$$

Now if $n > m$ and $n, m \in J$, we have $e_n \notin B_{n,m}$ and $e_m \notin B_{m,m}$. Therefore $e_m \notin B_{m,n}$. Condition (b) insures that $\|e_n - e_m\| \geq \theta$. We have thus found a sequence of points $(e_n)_{n\in J}$ in A, such that for every $n \neq m$, $n, m \in J$, $\|e_n - e_m\| \geq \theta > 1$ which contradicts $\delta(A) < 1$.

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