LOJASIEWICZ TYPE INEQUALITIES AND NEWTON DIAGRAMS

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ABSTRACT. We give an upper estimation for the Lojasiewicz exponent $\alpha_0(f)$ for a nondegenerate holomorphic function germ f using the Newton polygon of f .

1. INTRODUCTION

Let $f(x)$ be a holomorphic function defined near the origin $0 \in \mathbb{C}^n$ and $x = (x_1, \dots, x_n)$ a local coordinate system of $\mathbb{C}^n$ near 0. It is well known that f has an isolated singularity at 0 if and only if there are positive numbers α , C such that the following Lojasiewicz type inequality (L_α) holds near 0.

$$(L_\alpha) \quad \begin{aligned} & |\text{grad } f(x)| \geq C|x|^\alpha, \\ & \text{where } \text{grad } f(x) = \left(\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x) \right), \end{aligned}$$

and $||$ is the usual euclid norm. This inequality has appeared as a characterization of C^0 -sufficiency of jets.

(1.1) **Theorem** [Chang-Lu, 2]. *Let f be an analytic function near 0. If (L_α) holds near 0 for some $\alpha < r$, then $j^r f$ is a C^0 -sufficient jet in holomorphic functions.*

Originally this theorem was proved by T.-C. Kuo in the real case (see [4]). S. Koike pointed out to the author that the converse of this theorem is true (see [1]).

Let $\alpha_0(f)$ be the minimal number of α such that (L_α) holds near 0. In [6, 5] Lichtin used the Newton polygon of f to give an estimation of $\alpha_0(f)$ for $n = 2$. In this note, we give an estimation of $\alpha_0(f)$ using the Newton polygon of f for general n (Theorem (3.2)). The key step of our proof is to analyze a face of the Newton polygon of f that is neither compact nor coordinate.

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2. NEWTON POLYGON

(2.1) Let f be an analytic function near $0 \in \mathbb{C}^n$, and let $\sum_{\nu} a_{\nu} x^{\nu}$ be the Taylor expansion of f at 0. Set $\mathbb{R}_+ = \{x \in \mathbb{R} | x \geq 0\}$,

$\Gamma_+(f)$ = the convex hull of $\{\nu + \mathbb{R}_+^n | a_{\nu} \neq 0\}$,

$\Gamma(f)$ = union of compact faces of $\Gamma_+(f)$, and

$\Gamma^k(f) = \{k\text{-dimensional face of } \Gamma(f)\}$.

We call $\Gamma_+(f)$ (resp. $\Gamma(f)$) the *Newton polygon* of f (resp. the *Newton boundary* of f).

(2.2) Let $a = (a_1, \dots, a_n) \in \mathbb{R}_+^n$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{R}_+^n)^*$, where $(\mathbb{R}_+^n)^*$ is the dual space of $\mathbb{R}_+^n$. Set

$$\langle a, \alpha \rangle := a_1 \alpha_1 + \dots + a_n \alpha_n,$$

$$M(a) := \max\{a_1, \dots, a_n\},$$

$$m(a) := \min\{a_1, \dots, a_n\},$$

$$l(\alpha) := \min\{\langle a, \alpha \rangle | a \in \Gamma_+(f)\},$$

$$\gamma(\alpha) := \{a \in \Gamma_+(f) | \langle a, \alpha \rangle = l(\alpha)\}, \text{ and}$$

$$\Gamma^*(f) := (\mathbb{R}_+^n)^* / \sim,$$

where the equivalent relation $\alpha \sim \alpha'$ is defined by $\gamma(\alpha) = \gamma(\alpha')$. Naturally we can identify an equivalence class with a rational polyhedral convex cone $\sigma = \mathbb{R}_+ a^1(\sigma) + \dots + \mathbb{R}_+ a^k(\sigma)$ where $a^1(\sigma), \dots, a^k(\sigma)$ are minimal generators and $a^i(\sigma) \in \mathbb{Z}^n$ is primitive (= as small as possible) by requirement. For the sake of simplicity, we often abbreviate $a^i(\sigma)$ as a^i . For a rational polyhedral convex cone σ , we write $a(\sigma)$ for the matrix $(a^1(\sigma), \dots, a^k(\sigma))$. We say that σ is a k -simplex if $a^1(\sigma), \dots, a^k(\sigma)$ are linearly independent.

We say that a vector a supports a face γ of $\Gamma_+(f)$ if $\gamma = \gamma(a)$. We say that a face γ of $\Gamma_+(f)$ corresponds to a cone $\sigma = \mathbb{R}_+ a^1(\sigma) + \dots + \mathbb{R}_+ a^k(\sigma)$ if $\gamma = \gamma(\sigma)$, where $\gamma(\sigma) = \bigcap_{i=1}^k \gamma(a^i(\sigma))$. We say that a face γ of $\Gamma_+(f)$ is coordinate if γ is supported by some vector a with $l(a) = 0$.

(2.3) Define f_{γ} by $\sum_{\nu \in \gamma} a_{\nu} x^{\nu}$ for $\gamma \subset (\mathbb{R}_+^n)^*$. Note that f_{γ} is a polynomial if γ is compact. We say that f is *nondegenerate* if the equations $\partial f_{\gamma} / \partial x_1 = \dots = \partial f_{\gamma} / \partial x_n = 0$ have no common solutions on $x_1, \dots, x_n \neq 0$ for any compact face γ of $\Gamma_+(f)$.

(2.4) It is worth noting that $(x_k \partial f / \partial x_k)_{\gamma} = (x_k \partial / \partial x_k)(f_{\gamma})$ for $\gamma \subseteq (\mathbb{R}_+^n)^*$.

3. RESULT

(3.1) Let $V(f)$ denote the set of primitive vector a such that $\gamma(a)$ is an $(n-1)$ -dimensional face of $\Gamma_+(f)$. For a subset A of $V(f)$, we set

$$\varphi(A) := \max_{c_i \geq 0} \min_j \left\{ \sum_i c_i a_j^i / \sum_i c_i l(a^i) \right\}, \quad \text{where } A = \{a^1, \dots, a^k\}.$$

Obviously, $\varphi(A') \leq \varphi(A)$ if $A' \subset A$. Define a rational number $m_0(f)$ by the following formula:

$$m_0(f) = \max_{A \subset V(f)} \left\{ \varphi(A)^{-1} \left| \sum_{a \in A} \mathbb{R}_+ a \text{ is in a cone of } \Gamma^*(f) \text{ and } \varphi(A) \neq 0 \right| \right\}.$$

It is easy to see that $\varphi(\{a\})^{-1}$ is equal to the maximum of the length from the origin to the points in the intersection of the union of all axes and the hyperplane that is normal to a and contains $\gamma(a)$.

(3.2) **Theorem.** Let f be an analytic function near 0. Assume that f has an isolated singularity at 0 and is nondegenerate in the sense of (2.3). Then $\alpha_0(f) \leq m_0(f) - 1$.

(3.3) **Corollary.** Let f be as above, and let r be the smallest integer with $r > m_0(f) - 1$. Then $j^r f$ is a C^0 -sufficient jet.

(3.4) **Remark.** Theorem (3.2) asserts nothing new when the function f is convenient (meaning that the Newton polygon $\Gamma_+(f)$ meets all coordinate axes). (See [3].) When f is not convenient, however there is a new result.

(3.5) **Example.** Set $f(x_1, x_2, x_3) = x_1^5 + x_2^5 + x_2 x_3^5$. Then $\alpha_0(f) \leq m_0(f) - 1 = 25/4 - 1 = 21/4$. In this case it is easy to see that the equal mark holds. Defining g by $f + x_3^{100}$, we get $\alpha_0(g) = \alpha_0(f) = 21/4$ and $m_0(g) = 100$. Thus we cannot expect the equal mark to hold in all cases.

(3.6) **Example.** Set $f(x_1, x_2, x_3) = x_1(x_2^6 + x_3^6) + x_2(x_1^6 + x_3^6) + x_3(x_1^6 + x_2^6) + x_1^3 x_2^3 + x_2^3 x_3^3 + x_1^3 x_3^3$. Then $\alpha_0(f) \leq m_0(f) - 1 = 15/2 - 1 = 6 + 1/2$. Hence the 7-jet of f is C^0 -sufficient. (See Figure 1.)

FIGURE 1

(3.7) **Example.** Set $f(x_1, x_2, x_3, x_4) = x_1^2 x_2^2 x_3^2 x_4^2 + x_1 x_3^8 + x_2 x_4^8 + x_1^8 x_4 + x_2^8 x_3$. By elementary computation, we get $m_0(f) - 1 = 910/94 - 1 = 8 + 32/47$. By our theorem, $\alpha_0(f) \leq 8 + 32/47$ and hence, the 9-jet of f is C^0 -sufficient.

(3.8) **Example.** Set $f(x_1, x_2, x_3, x_4) = x_1^2 x_2^2 x_3^2 x_4^2 + x_1 x_3^8 + x_2 x_4^8 + x_1^8 x_4 + x_2^8 x_3 + x_2^4 x_3^4$. By elementary computation, we get $m_0(f) - 1 = 910/94 - 1 = 8 + 32/47$. By our theorem, $\alpha_0(f) \leq 8 + 32/47$.

4. PROOF

(4.1) We first prepare some notations. Let σ denote a rational polyhedral convex cone in the first quadrant;

$$\sigma = \mathbb{R}_+ a^1(\sigma) + \cdots + \mathbb{R}_+ a^k(\sigma),$$

as in (2.2). Let

$$\begin{aligned} V(\sigma) &= \{a^1(\sigma), \dots, a^k(\sigma)\}, \\ V_+(\sigma) &= \{a \in V(\sigma) | a_i > 0 \text{ for any } i\}, \\ V_-(\sigma) &= V(\sigma) \cap \{e_1, \dots, e_n\}, \\ V_0(\sigma) &= V(\sigma) - V_+(\sigma) - V_-(\sigma), \\ N(\sigma) &= \{i | a_i \neq 0 \text{ for some } a \in V_0(\sigma)\}, \\ M(\sigma) &= \{i | a_i = l(a) \text{ for any } a \in V_0(\sigma)\}, \text{ and} \\ N_-(\sigma) &= \{i | e_i \in V_-(\sigma)\}. \end{aligned}$$

Here e_i denote the vector $(0, \dots, 0, \overset{i}{1}, 0, \dots, 0)$.

Using the identification in (2.2), we consider $\Gamma^*(f)$ to be a rational polyhedral finite subdivision of the first quadrant.

We first prove some lemmata.

(4.2) **Lemma.** Let σ be in a cone of $\Gamma^*(f)$ with $V_0(\sigma) \neq \emptyset$. If $l(a) > 0 \forall a \in V_0(\sigma)$, then we can write $f(x) = \sum_{i \in N(\sigma)} x_i f_i(x)$, for some power series $f_i(x)$.

Proof. Since $l(a) > 0 \forall a \in V_0(\sigma)$, we have $V_-(\sigma) = \{a \in V(\sigma) | l(a) = 0\}$. If there is a $\nu \in \Gamma_+(f)$ with $\nu_i = 0$ for any $i \in N(\sigma)$, then $\langle \nu, a \rangle = 0$ for any $a \in V_0(\sigma)$. This is a contradiction. Q.E.D.

Let h_i denote the restriction of f_i to the set $\{x_j = 0 | j \in N(\sigma)\}$, and $L(\sigma)$ the set of i 's such that h_i is not identically zero.

(4.3) **Lemma.** Suppose that σ is in a cone of $\Gamma^*(f)$ with $V_0(\sigma) \neq \emptyset$. If f defines an isolated singularity at 0, then $\#L(\sigma) \geq n - \#N(\sigma)$.

Proof. Since f has an isolated singularity at 0, we obtain $\{h_i = 0 | i \in N(\sigma)\} = \{0\}$ on $\{x_j = 0 | j \in N(\sigma)\}$. In particular, $\#L(\sigma) \geq n - \#N(\sigma)$. Q.E.D.

Remark. Note that if $\gamma(a) \notin \{\nu_i = 0\}$, then $l(a) \geq a_i$, for any $i = 1, \dots, n$. Thus if $\gamma(a) \notin \{\nu_1, \dots, \nu_n = 0\}$, then $l(a) \geq M(a) \forall a \in V_0(\sigma)$.

(4.4) **Lemma.** Suppose that σ is in a cone of $\Gamma^*(f)$ with $V_0(\sigma) \neq \emptyset$ and that $l(a) \geq M(a) \quad \forall a \in V_0(\sigma)$. Then $L(\sigma) \subset M(\sigma) \subset N(\sigma)$.

Proof. Let $i \in L(\sigma)$. If we write $h_i = \sum c_{i,\nu} x^\nu$ (a power series in variable x_k with $k \notin N(\sigma)$), then $l(a) \leq \min\{\langle \nu, a \rangle | c_{i,\nu} \neq 0\} + a_i = a_i \leq M(a) \leq l(a)$, for any $a \in V_0(\sigma)$. Thus $i \in M(\sigma)$. Q.E.D.

(4.5) **Lemma.** Suppose that σ is in a cone of $\Gamma^*(f)$ with $V_0(\sigma) \neq \emptyset$, and that $l(a) \geq M(a) \quad \forall a \in V_0(\sigma)$. If f defines an isolated singularity at 0 and $N(\sigma)$ is a proper subset of $\{1, \dots, n\}$, then an integral point $\nu = (\nu_1, \dots, \nu_n)$ of the face corresponding to σ has one of the following properties:

- (a) There is a number $k \in M(\sigma)$ such that $\nu_k = 1$ and $\nu_m = 0$ for any $m \in N(\sigma) - \{k\}$.
- (b) $\nu_k = 0$ for any $k \in M(\sigma)$.

Proof. From the proof of Lemma (4.4), there is an integral point ν in $\Gamma_+(f)$ with $l(a) = \langle a, \nu \rangle = M(a)$ for $a \in V_0(\sigma)$. Thus the lemma holds. Q.E.D.

Let $\gamma'(\sigma)$ denote the convex hull of integral points of type (a) in the set $\{\nu | \langle a, \nu \rangle = l(a) \quad \forall a \in V_+(\sigma) \cup V_0(\sigma)\}$.

(4.6) **Lemma.** Let σ be in a cone of $\Gamma^*(f)$ with $\gamma'(\sigma) \neq \emptyset$. Assume that f defines an isolated singularity at 0, and $l(a) \geq M(a)$, $\forall a \in V_0(\sigma)$. If $N(\sigma) \cup N_-(\sigma) \neq \{1, \dots, n\}$, and if $N(\sigma) \not\supset N_-(\sigma)$, then the face $\gamma'(\sigma)$ intersects with the set $\{\nu_j = 0 | j \in N_-(\sigma) - N(\sigma)\}$.

Proof. Since $N(\sigma)$ is a proper subset of $\{1, \dots, n\}$, and by (4.4), we have $L(\sigma) \neq \emptyset$. After a suitable renumbering, we may assume $M(\sigma) = \{1, \dots, s\}$, $N(\sigma) = \{1, \dots, s, \dots, c\}$, and $N_-(\sigma) - N(\sigma) = \{c+1, \dots, d\}$.

By the preceding lemmata, we write

$$\begin{aligned} f(x) &= \sum_{i=1}^c x_i f_i(x_1, \dots, x_n) \\ &= \sum_{i=1}^s x_i \{h_i(x_{c+1}, \dots, x_n) + \sum_{j=1}^c x_j h_{ij}(x_1, \dots, x_n)\} \\ &\quad + \sum_{i=s+1}^c x_i \sum_{j=1}^c x_j h_{ij}(x_1, \dots, x_n) \\ &= \sum_{i=1}^s x_i h_i(x_{c+1}, \dots, x_n) + \sum_{i,j=1}^c x_i x_j h_{ij}(x_1, \dots, x_n), \end{aligned}$$

where h_{ij} is some power series.

Assume that $\gamma'(\sigma) \cap \{x_j = 0 | j \in N_-(\sigma) - N(\sigma)\}$ is empty. Then we can write

$$h_i(x_{c+1}, \dots, x_n) = \sum_{j=c+1}^d x_j g_{ij}(x_{c+1}, \dots, x_n),$$

where g_{ij} is some power series in $(n-c)$ variables $x_{c+1}, \dots, x_n$. We thus have $\partial f / \partial x_i = \sum_{j=c+1}^d x_j g_{ij}$ ($1 \leq i \leq s$); 0 ($s+1 \leq i \leq n$) on $\{x_1 = \dots = x_c = 0\}$. Because f defines an isolated singularity at 0 , we obtain $d = c$ or $d = n$. These contradict the suppositions of the lemma. Q.E.D.

Lemma. Suppose that f has an isolated singularity at 0 , and that σ is in a cone of $\Gamma^*(f)$. Suppose that a^1 and a^2 are in $V_0(\sigma)$ with $\gamma(a^2) \notin \{\prod_{i \in L(\mathbb{R}_+ a^2)} \nu_i = 0\}$. If $N(\mathbb{R}_+ a^1) \subset N(\mathbb{R}_+ a^2)$, then $\gamma'(a^1) \supset \gamma'(a^2)$.

Proof. It is enough to show that $L(\mathbb{R}_+ a^1) \supset L(\mathbb{R}_+ a^2)$. If $i \in L(\mathbb{R}_+ a^2)$, then there is a point $\nu = (\nu_1, \dots, \nu_n)$ in $\gamma'(a^2)$ such that $\nu_i = 1$ and $\nu_j = 0$ $\forall j \in N(\mathbb{R}_+ a^2) - \{i\}$.

Since

$$l(a^1) \leq \langle a^1, \nu \rangle = \sum_{j \in N(\mathbb{R}_+ a^1)} a_j^1 \nu_j = \begin{cases} a_i^1 & \text{if } i \in N(\mathbb{R}_+ a^1), \\ 0 & \text{if } i \notin N(\mathbb{R}_+ a^1), \end{cases}$$

we obtain that $i \in L(\mathbb{R}_+ a^1)$. Q.E.D.

Let Σ be a finite collection of rational polyhedral convex cones in the first quadrant. We set V the set of integral primitive vectors whose nonnegative multiplications are precisely 1-cones of Σ ; V_+ the subset of V , whose element supports a compact face of $\Gamma_+(f)$; V_- the subset of V , whose element supports a coordinate face of $\Gamma_+(f)$; and $V_0 = V - V_+ - V_-$.

(4.7) **Proposition.** Assume that f defines an isolated singularity at 0 . Then, we can construct a finite collection Σ of rational polyhedral convex cones that satisfies the following conditions:

- (1) Each element of Σ is a simplex.
- (2) The union of the cones of Σ is the first quadrant.
- (3) Any face of a cone in Σ belongs to Σ .
- (4) $\forall \sigma \in \Sigma$, there is some cone τ of $\Gamma^*(f)$ with $\sigma \subset \tau$.
- (5) $\gamma(a) \notin \{\nu_1, \dots, \nu_n = 0\} \quad \forall a \in V_0$.
- (6) $\gamma'(\sigma)$ is not empty for any σ in Σ .
- (7) $N(\sigma) \cup N_-(\sigma)$ is a proper subset of $\{1, \dots, n\}$ for any cone σ of Σ .

(4.8) **Proposition.** Suppose that f has an isolated singularity at 0 and is non-degenerate in the sense of (2.3). If Σ satisfies conditions (1)–(7), then we have

$$\alpha_0(f) \leq \max\{l(a)/m(a) | a \in V_+\} - 1.$$

These two propositions give an upper estimate of the Lojasiewicz exponent $\alpha_0(f)$. By the proof of Proposition (4.7) below, we obtain our theorem (3.2). The proof of Proposition (4.8) will start at (4.10) and conclude at (4.17).

Proof of Proposition (4.7). First we remark that $\Gamma^*(f)$ satisfies conditions (2)–(5). Let σ be a cone of $\Gamma^*(f)$. Let A_i ($i = 1, \dots, s$) be a subset of $V_0(\sigma)$ satisfying the following conditions:

- (a1) $V_0(\sigma)$ is the disjoint union of A_i 's ($i = 1, \dots, s$).
- (a2) $a, b \in A_i$ if and only if there is a finite sequence $\{a_i\}_{i=1, \dots, t}$ in A_i such that $a_1 = a$, $a_t = b$, and $\mathbb{R}_+ a^i + \mathbb{R}_+ a^{i+1}$ is a 2-face of σ for $i = 1, \dots, t-1$.

Let $V_0^b(\sigma)$ be the set of vector a in $V_0(\sigma)$ such that $\gamma(\sigma)$ is a point in $\gamma(a)$ of type (b) in Lemma (4.5).

Step 1. We construct a finite collection Σ_1 satisfying conditions (2)–(5), and the following:

- (8) $N(\sum_{a \in A_i} \mathbb{R}_+ a) \neq \{1, \dots, n\} \quad \forall i, \sigma$.

Let σ be a cone of $\Gamma^*(f)$ with $N(\sum_{a \in A_i} \mathbb{R}_+ a) = \{1, \dots, n\}$ for some i . Set $b = \sum_{a \in A_i} c_a a$ ($c_a > 0$) with $\varphi(\{b\}) = \varphi(A_i)$. Replace σ by the cones $\mathbb{R}_+ b + \sum_{a \in A} \mathbb{R}_+ a$ (A is a subset of $V(\sigma)$), of which the 1-cone $\mathbb{R}_+ b$ is a face. Then the cone $\sum_{a \in A_i} \mathbb{R}_+ a$ is not a face of these cones. Repeating this procedure, we can construct Σ_1 satisfying conditions (2)–(5), (8).

In many examples (e.g. (3.5)–(3.8)), we can skip Step 1, set $\Sigma_1 = \Gamma^*(f)$, and proceed as follows:

Step 2. In this step, we construct a finite collection Σ_2 satisfying conditions (2)–(6). Our construction is based on the following lemma that is intuitively trivial.

Lemma. Let $\sigma = \mathbb{R}_+ a^1 + \dots + \mathbb{R}_+ a^k$ be a cone and $a = a^1$. Suppose that $\mathbb{R}_+ a^1 + \mathbb{R}_+ a^i$ is a 2-face of σ if and only if $i \in \{2, \dots, s\}$. Then $\sigma = \sigma_a \cup \sigma'_a$, where $\sigma_a = \mathbb{R}_+ a^1 + \dots + \mathbb{R}_+ a^s$ and $\sigma'_a = \mathbb{R}_+ a^2 + \dots + \mathbb{R}_+ a^k$.

Let σ be a cone of Σ_1 with $\gamma'(\sigma) = \emptyset$. Let $a \in V_0^b(\sigma)$ be a vector such that there are no $b \in V_0^b(\sigma)$ with $N(\mathbb{R}_+ b) \supsetneq N(\mathbb{R}_+ a)$. By our construction and (4.3), we have $\gamma'(\sigma_a) \neq \emptyset$. Repeating this procedure for σ'_a , we construct a finite collection Σ_2 satisfying conditions (2)–(6), because of condition (8) and (4.3).

Step 3. We construct a finite collection Σ_3 satisfying conditions (1)–(6). If $\sigma = \mathbb{R}_+ a^1 + \dots + \mathbb{R}_+ a^k$, a cone of Σ_2 , is not a simplex, then replace σ by the cones $\mathbb{R}_+ a^{i_1} + \dots + \mathbb{R}_+ a^{i_p}$ ($1 \leq i_1 < \dots < i_p \leq k$) which are p -simplices.

Step 4. We construct a finite collection Σ satisfying conditions (1)–(7). Let $\sigma = \mathbb{R}_+ a^1 + \dots + \mathbb{R}_+ a^n$ be an n -simplex in Σ_3 with $N(\sigma) \cup N_-(\sigma) = \{1, \dots, n\}$. Set $b = \sum_{a \in V_0(\sigma)} a + \sum_{i \in N_-(\sigma)} e_i$. (Note that $\varphi(\{b\}) = 1$.)

Replace σ by the cones $\sigma_i = \mathbb{R}_+ b + \sum_{j \neq i} \mathbb{R}_+ a^j$ ($i = 1, \dots, n$). Since $N(\sigma_i) \cup N_-(\sigma_i) \neq \{1, \dots, n\}$, we can construct a finite collection Σ satisfying conditions (1)–(7). Q.E.D.

We construct a collection Σ concretely for some of the preceding examples.

FIGURE 2

We choose a simplicial subdivision Σ of $\Gamma^*(f)$ for these cases. In Example (3.6), $\Gamma^*(f)$ is determined by its plane section with the plane $\alpha_1 + \alpha_2 + \alpha_3 = 1$. (See the solid lines in Figure 2.) Adding the dotted lines in Figure 2, for example, we get a simplicial subdivision Σ of $\Gamma^*(f)$.

In Example (3.8), $\Gamma^*(f)$ is determined by its hyperplane section with the hyperplane $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$. (See Figure 3.) The data of polyhedral cones are given in Table 1.

We construct a simplicial subdivision Σ of $\Gamma^*(f)$ that contains at least the following simplices:

$$\begin{aligned} & a^1 a^7 a^{12} a^{14}, \quad a^1 a^7 a^{14} a^{16}, \quad a^2 a^8 a^{12} a^{17}, \quad a^2 a^8 a^{15} a^{17}, \quad a^3 a^9 a^{14} a^{15}, \\ & a^3 a^9 a^{11} a^{15}, \quad a^4 a^{10} a^{16} a^{17}, \quad a^4 a^{10} a^{13} a^{16}, \quad a^6 a^{11} a^{13} a^{15}, \quad a^3 a^6 a^{11} a^{15}. \end{aligned}$$

Here we denote $a^i a^j a^k a^l = \mathbb{R}_+ a^i + \mathbb{R}_+ a^j + \mathbb{R}_+ a^k + \mathbb{R}_+ a^l$.

Let $\mathbb{C}^n(\sigma)$ be a copy of $\mathbb{C}^n$ for each n -cone σ of Σ , and let $y_\sigma := (y_{\sigma,1}, \dots, y_{\sigma,n})$ be a coordinate system of $\mathbb{C}^n(\sigma)$. For a matrix $A = (a_i^j) \in \text{Mat}(n, n; \mathbb{Z})$ and $y = (y_1, \dots, y_n)$, set

$$Ay = (y_1^{a_1^1}, \dots, y_n^{a_n^1}, \dots, y_1^{a_1^n}, \dots, y_n^{a_n^n}).$$

Define the mapping $\pi_\sigma: \mathbb{C}^n(\sigma) \rightarrow \mathbb{C}^n$ by $\pi_\sigma(y_\sigma) := {}^{a(\sigma)}y_\sigma$. Set

$$\begin{aligned} W_\sigma &:= \{y_\sigma \in \mathbb{C}^n(\sigma) \mid |y_{\sigma,j}| \leq 1, \text{ for } j = 1, \dots, n\}, \text{ and} \\ V &:= \{x \in \mathbb{C}^n \mid |x_i| \leq 1, \text{ for } i = 1, \dots, n\}. \end{aligned}$$

Let W_Σ be the disjoint union of W_σ for n -cones σ of Σ . Define the mapping $\pi_\Sigma: W_\Sigma \rightarrow V$ by $\pi_\Sigma(y_\sigma) := \pi_\sigma(y_\sigma)$ for $y_\sigma \in W_\sigma$. For a subset I of $\{1, \dots, n\}$,

FIGURE 3. The image of $\Gamma^*(f) \cap \{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1\}$ under the projection defined by $x = 1.4\alpha_1 + 2\alpha_2 + 2.6\alpha_4$; $y = 3\alpha_1 - 0.6\alpha_2 + \alpha_4$.

set

$$E_{\sigma, I} = \{y_{\sigma} \in W_{\sigma} \mid y_{\sigma, i} = 0, \text{ for any } i \in I\}, \text{ and}$$

$$E_{\sigma, I}^* := \{y_{\sigma} \in E_{\sigma, I} \mid y_{\sigma, j} \neq 0, \text{ for any } j \in \{1, \dots, n\} - I\}.$$

The following lemma makes us consider the pullback of the inequality (L_{α}) via π_{Σ} .

(4.9) **Lemma.** (1) $\pi_{\sigma}^{-1}(0) \cap W_{\sigma}$ is compact. (2) π_{σ} is surjective.

Proof. (1) Because $\pi_{\sigma}^{-1}(0) \cap W_{\sigma}$ is the intersection of a union of some coordinate spaces with the bounded closed set W_{σ} , (1) is obvious.

(2) For any $x \in V$, set

$$x_i = r_i \cdot e^{2\pi\sqrt{-1}\theta_i}, \quad r_i \geq 0, \quad 0 \leq \theta_i < 1.$$

Set

$$y_{\sigma, j} = r_{\sigma, j} \cdot e^{2\pi\sqrt{-1}\theta_{\sigma, j}}, \quad r_{\sigma, j} \geq 0, \quad 0 \leq \theta_{\sigma, j} < 1.$$

Since

$$x_i = y_{\sigma, 1}^{a_i^1(\sigma)} \cdots y_{\sigma, n}^{a_i^n(\sigma)},$$

we get

$$(4.9.1) \quad r_i = r_{\sigma, 1}^{a_i^1(\sigma)} \cdots r_{\sigma, n}^{a_i^n(\sigma)},$$

TABLE 1

$(1) \setminus (2)$	$\begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 8 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 8 \end{pmatrix}$	$\begin{pmatrix} 8 \\ 0 \\ 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 8 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$
$a^1 = (102, 158, 101, 94)$	*	*	*	*		
$a^2 = (4, 4, 3, 3)$	*	*	*			*
$a^3 = (24, 31, 25, 32)$	*	*		*		*
$a^4 = (101, 94, 158, 102)$	*		*	*	*	
$a^5 = (31, 24, 32, 25)$	*		*		*	*
$a^6 = (3, 3, 4, 4)$	*			*	*	*
$a^7 = (8, 64, 7, 0)$		b	*	b		
$a^8 = (32, 8, 0, 3)$		*	b			b
$a^9 = (0, 1, 1, 8)$		b		*		b
$a^{10} = (7, 0, 64, 8)$		b	b	*		
$a^{11} = (0, 3, 4, 28)$				*	b	b
$a^{12} = (1, 1, 0, 0)$		*	*			
$a^{13} = (0, 0, 1, 1)$				*	*	
$a^{14} = (0, 1, 0, 0)$		*		*		
$a^{15} = (0, 0, 0, 1)$		*			*	*
$a^{16} = (0, 0, 1, 0)$			*	*		
$a^{17} = (1, 0, 0, 0)$			*		*	*

(1) = the vectors supporting 3-faces of $\Gamma_+(f)$. (2) = the vertices of $\Gamma_+(f)$.

and

$$(4.9.2) \quad \theta_i \equiv a_i^1(\sigma)\theta_{\sigma,1} + \cdots + a_i^n(\sigma)\theta_{\sigma,n} \pmod{\mathbb{Z}}.$$

Since $a(\sigma)$ has the maximal rank, (4.9.2) has a solution. We have to solve (4.9.1) for some σ under the condition $0 \leq r_{\sigma,j} \leq 1$. We may assume that $r_i \neq 0$ for $i = 1, \dots, n$. Thus we obtain that

$$“(4.9.1) \Leftrightarrow \log r_i = \sum_{j=1}^n a_i^j(\sigma) \log r_{\sigma,j} \text{ on } r_{\sigma,1} \cdots r_{\sigma,n} \neq 0.”$$

Therefore “ $(-\log r_1, \dots, -\log r_n)$ is in some $\sigma \Leftrightarrow (-\log r_{\sigma,1}, \dots, -\log r_{\sigma,n}) \in$ the first quadrant $\Leftrightarrow r_{\sigma,j} \leq 1$ for $j = 1, \dots, n$.” Because of condition (2), σ and $r_{\sigma,j}$ satisfying (4.9.1). Since $(\mathbb{C} - 0)^n \cap V$ is dense in V , and in view of (1), (4.9.1) has a solution with $r_{\sigma,j} \leq 1$. Q.E.D.

(4.10) We fix an n -simplex σ and consider the pullback of the inequality (L_α) via π_σ . It is enough to show that the pullback inequality holds on W_σ because of (4.9). For the sake of simplicity, we often omit the letter σ when the case is clear. For example, $a_i^j = a_i^j(\sigma)$, $a^j = a^j(\sigma)$, $y = y_\sigma$, $y_j = y_{\sigma,j}$, $E_I = E_{\sigma,I}$, $E_I^* = E_{\sigma,I}^*$, and so on.

We prepare some notations. Set $x = {}^A y$, where $A = (a_i^j)$.

$$N_+ := \{j \in \{1, \dots, n\} | a_i^j > 0, \text{ for any } i\},$$

$$N_- := \{j \in \{1, \dots, n\} | l(a^j) = 0\}.$$

$$N_0 := \{1, \dots, n\} - N_+ - N_-.$$

For a subset I of $\{1, \dots, n\}$, set

$$I_+ := I \cap N_+, \quad I_0 := I \cap N_0, \quad I_- := I \cap N_-,$$

$$N_I := \{i | a_i^j \neq 0, \text{ for some } j \in I_0\}, \text{ and}$$

$$M_I := \{i | a_i^j = l(a^j), \text{ for any } j \in I_0\}.$$

Set $e_i = (0, \dots, 0, \overset{i}{1}, 0, \dots, 0)$, and $\gamma_I = \bigcap_{j \in I} \gamma(a^j)$. It is easy to see that γ_{I_+} (resp. γ_{I_-} , γ_{I_0}) is a compact (resp. coordinate, noncompact and noncoordinate) face of $\Gamma_+(f)$.

Lemma. $\pi(E_I) = \{0\}$ if and only if $\bigcap_{i \in I} \gamma(a^i)$ is a compact face of $\Gamma_+(f)$. (It is easy to see and we omit the proof.)

In order to analyze the pullback of (L_α) , define $g_k(y_\sigma)$ and $g'_k(y_\sigma)$ by

$$\left(x_k \cdot \frac{\partial f}{\partial x_k}\right)(\pi_\sigma(y_\sigma)) = \prod_{j=1}^n y_{\sigma,j}^{l(a^j(\sigma))} \cdot g_k(y_\sigma),$$

and

$$\frac{\partial f}{\partial x_k}(\pi_\sigma(y_\sigma)) = \prod_{j \in N_+} y_{\sigma,j}^{l(a^j(\sigma)) - a_k^j(\sigma)} \cdot g'_k(y_\sigma).$$

Then

$$g_k(y_\sigma) = \sum_\nu \nu_k a_\nu y_{\sigma,1}^{\langle \nu, a^1 \rangle - l(a^1)} \dots y_{\sigma,n}^{\langle \nu, a^n \rangle - l(a^n)},$$

$$g'_k(y_\sigma) = \sum_\nu \nu_k a_\nu \prod_{j \in N_+} y_{\sigma,j}^{\langle \nu, a^j \rangle - l(a^j)} \cdot \prod_{j \in N_0 \cup N_-} y_{\sigma,j}^{\langle \nu, a^j \rangle - a_k^j}.$$

Note that g_k and g'_k are analytic functions. The key step of our proof for (4.8) is to prove the nonvanishing of $g' := \sum |g'_k(y_\sigma)|^2$ on $\pi_\sigma^{-1}(0)$. We will complete this step at (4.16.1).

(4.11) Since

$$\prod_{j \in N_0 \cup N_-} y_{\sigma,j}^{a_k^j(\sigma)} \cdot g'_k(y_\sigma) = \prod_{j \in N_0 \cup N_-} y_{\sigma,j}^{l(a^j(\sigma))} \cdot g_k(y_\sigma),$$

we get $\{g'_1 = \dots = g'_n = 0\} = \{g_1 = \dots = g_n = 0\}$, on $\prod_{j \in N_0 \cup N_-} y_{\sigma,j} \neq 0$. Thus, if $\prod_{j \in N_0 \cup N_-} y_{\sigma,j} \neq 0$, then the positivity of g' follows from the positivity of $g := \sum |g_k|^2$, which is easy to show (Lemma (4.12)). In the case $\prod_{j \in N_0 \cup N_-} y_{\sigma,j} = 0$, we show the positivity of g' directly in (4.16) using the properties of noncompact and noncoordinate faces, that were described above.

(4.12) **Lemma.** For any $y \in E_I^*$, the following conditions are equivalent:

- (1) $g_k(y) = 0$;
- (2) $\partial f_{\gamma_I} / \partial x_k(\tilde{x}) = 0$,

where $\tilde{x} = {}^A \tilde{y}$, $\tilde{y} = (\tilde{y}_1, \dots, \tilde{y}_n)$, $\tilde{y}_j = 1$, if $j \in I$; $\tilde{y}_j = y_j$, otherwise.

Proof. Since

$$\begin{aligned} g_k|E_I &= \sum_{\nu \in \gamma_I} \nu_k \cdot a_\nu \prod_{j \in I} y_j^{\langle \nu, a^j \rangle - l(a^j)} \\ &= \sum_{\nu \in \gamma_I} \nu_k \cdot a_\nu \prod_{j=1}^n \tilde{y}_j^{\langle \nu, a^j \rangle - l(a^j)}, \end{aligned}$$

we obtain “ $g_k(y) = 0$ for $y \in E_I^* \Leftrightarrow \sum_{\nu \in \gamma_I} \nu_k \cdot a_\nu \prod_{j=1}^n \tilde{y}_j^{\langle \nu, a^j \rangle} = 0 \Leftrightarrow \sum_{\nu \in \gamma_I} \nu_k \cdot a_\nu \tilde{x}^\nu = 0 \Leftrightarrow \partial f_{\gamma_I} / \partial x_k(\tilde{x}) = 0$.”

Q.E.D.

(4.13) **Lemma.** For any $y \in E_I^*$, the following conditions are equivalent:

- (1) $g'_k(y) = 0$;
- (2) $\partial f_{\Gamma_{I,k}} / \partial x_k(\tilde{x}) = 0$,

where $\Gamma_{I,k} := \{\nu \in \Gamma_+(f) | \langle \nu, a^j \rangle = l(a^j), \text{ for any } j \in I_+; \langle \nu, a^j \rangle = a_k^j, \text{ for any } j \in I_0 \cup I_-\}$.

Proof. Since

$$\begin{aligned} g'_k|E_I &= \sum_{\nu \in \Gamma_{I,k}} \nu_k \cdot a_\nu \prod_{j \in N_+ - I_+} y_j^{\langle \nu, a^j \rangle - l(a^j)} \prod_{j \in N_0 \cup N_- - I_0 \cup I_-} y_j^{\langle \nu, a^j \rangle - a_k^j} \\ &= \sum_{\nu \in \Gamma_{I,k}} \nu_k \cdot a_\nu \prod_{j \in N_+} \tilde{y}_j^{\langle \nu, a^j \rangle - l(a^j)} \prod_{j \in N_0 \cup N_-} \tilde{y}_j^{\langle \nu, a^j \rangle - a_k^j} \\ &= \sum_{\nu \in \Gamma_{I,k}} \nu_k \cdot a_\nu \prod_{j=1}^n \tilde{y}_j^{\langle \nu, a^j \rangle - l(a^j)} \cdot \prod_{j \in N_0 \cup N_-} \tilde{y}_j^{l(a^j) - a_k^j}, \end{aligned}$$

we obtain

$$“g'_k(y) = 0 \text{ for } y \in E_I^* \Leftrightarrow \sum_{\nu \in \Gamma_{I,k}} \nu_k \cdot a_\nu \prod_{j=1}^n \tilde{y}_j^{\langle \nu, a^j \rangle} = 0 \Leftrightarrow \partial f_{\Gamma_{I,k}} / \partial x_k(\tilde{x}) = 0.”$$

Q.E.D.

(4.14) **Lemma.** Assume that f has an isolated singularity at 0, and that Σ satisfies conditions (1)–(7).

- (1) If $k \notin M_I$, then $\Gamma_{I,k} = \emptyset$.
- (2) If $k \in M_I$, then $\frac{\partial}{\partial x_k}(f_{\Gamma_{I,k}}) = \frac{\partial}{\partial x_k}(f_{\tilde{\gamma}_I})$,

where $\tilde{\gamma}_I = \gamma_{I_+ \cup I_0} \cap \delta_{S_I - N_I} \cap \gamma(a')$,

$$\delta_J = \{\nu_j = 0 | j \in J\}, \quad J \subset \{1, \dots, n\},$$

$S_I = \{j \in \{1, \dots, n\} \mid \text{there is a number } i \in I_- \text{ with } a^i = e_j\}$, and a' is a vector with $f_{\gamma(a')} = \sum_{i \in M_I} x_i h_i$ (see the sentence before Lemma (4.3) for the definition of h_i).

Note. It is easy to see that if we set that $a'_i = 1$, if $i \in M_I$; 2, if $i \in N_I - M_I$; 0, otherwise, then $f_{\gamma(a')} = \sum_{i \in M_I} x_i h_i$. (Use the fact that $l(a') = 1$.) Also note that $\tilde{\gamma}_I$ does not depend on $k \in M_I$.

Remark. By Lemma (4.6), conditions (5)–(7) imply $\tilde{\gamma}_I = \gamma_{I_+ \cup I_0} \cap \gamma(a')$ is not empty.

Proof. (1) By the definition of l , $\Gamma_{I,k}$, and g'_k , (1) is obvious.

(2) By the definition of g'_k , we obtain

$$\begin{aligned} \Gamma_{I,k} &= \{\nu \in \Gamma_+(f) \mid \langle \nu, a^j \rangle = l(a^j) \text{ for } j \in I_+ \cup I_0; \langle \nu, a^j \rangle = a_k^j \text{ for } j \in I_-\} \\ &= \gamma_{I_+ \cup I_0} \cap \delta_{S_I - \{k\}} \cap \{\nu_k = 1 \mid \text{if } k \in S_I\}. \end{aligned}$$

Since $k \in M_I$ and by (4.4), $\nu_k \leq 1$ if $\nu \in \Gamma_{I,k}$. Differentiating by x_k , the point $\nu \in \Gamma_{I,k}$ with $\nu_k = 0$ does not contribute to $(\partial/\partial x_k)f_{\Gamma_{I,k}}$. Thus we obtain $(\partial/\partial x_k)f_{\Gamma_{I,k}} = (\partial/\partial x_k)(f_{\gamma_{I_+ \cup I_0} \cap \delta_{S_I - \{k\}} \cap \{\nu_k = 1\}})$. For a suitable choice of f_i 's in (4.2), we have

$$\begin{aligned} f_{\gamma_{I_+ \cup I_0} \cap \Delta_{+S_I - \{k\}} \cap \{\nu_k = 1\}} &= \left(\sum_{i=1}^n x_i f_i \right) \{\nu_k = 1\} \cap \gamma_{I_+ \cup I_0} \cap \delta_{S_I - \{k\}} \\ &= (x_k(f_k)_{\delta_{N_I}}) \gamma_{I_+ \cup I_0} \cap \delta_{S_I - N_I} \\ &= (x_k h_k) \gamma_{I_+ \cup I_0} \cap \delta_{S_I - N_I}. \end{aligned}$$

Because the monomial in $x_j h_j$ for $j \in M_I - \{k\}$ does not contribute to $(\partial/\partial x_k)(\sum_{j \in M_I} x_j h_j)$, we have

$$\frac{\partial}{\partial x_k}(x_k h_k) = \frac{\partial}{\partial x_k} \left(\sum_{j \in M_I} x_j h_j \right) = \frac{\partial}{\partial x_k}(f_{\gamma(a')}).$$

By (2.4), we therefore, obtain

$$\frac{\partial}{\partial x_k}(f_{\Gamma_{I,k}}) = \frac{\partial}{\partial x_k}(f_{\gamma(a')} \cap \gamma_{I_+ \cup I_0} \cap \delta_{S_I - N_I}). \quad \text{Q.E.D.}$$

(4.15) **Lemma.** Assume that f is nondegenerate in the sense of (2.3), f has an isolated singularity at 0, and that $I_0 \neq \emptyset$. Then

$$\{\partial f_\gamma / \partial x_k = 0, \text{ for any } k \in M_I\} \subset \{x_1 \cdots x_n = 0\}, \quad \text{where } \gamma = \tilde{\gamma}_I.$$

Proof. We show the contrapositive. For the sake of simplicity, we assume that $L_I = \{1, \dots, s\}$, $N_I = \{1, \dots, s, s+1, \dots, c\}$. By (4.3), we get that $s \geq 1$, and that $L_I \subset M_I$. Because of the proof of (4.14), we have

$$f_\gamma(x) = \sum_{k=1}^s x_k H_k(x_{c+1}, \dots, x_n), \quad \text{where } H_k = \partial f_\gamma / \partial x_k.$$

Assume that there is a $(x_{c+1}^0, \dots, x_n^0)$ such that $x_{c+1}^0 \cdots x_n^0 \neq 0$, and that $H_k(x_{c+1}^0, \dots, x_n^0) = 0$, for any $k \in M_I$. By the assumption of nondegeneracy of subfaces of γ ,

$$\left\{ \sum_{j \in J} x_j \cdot \frac{\partial H_j}{\partial x_i}(x_{c+1}^0, \dots, x_n^0) = 0 \text{ for } i = c+1, \dots, n \right\} \subset \left\{ \prod_{j \in J} x_j = 0 \right\},$$

for any subset J of L_I . Thus we get $\text{rank}((\partial H_j / \partial x_i)(x_{c+1}^0, \dots, x_n^0)) = s$. On the contrary, because $H_1, \dots, H_s$ are weighted homogeneous polynomials for some weight $(a_{c+1}, \dots, a_n)$,

$$\sum_{i=c+1}^n a_i \cdot x_i^0 \cdot \frac{\partial H_j}{\partial x_i}(x_{c+1}^0, \dots, x_n^0) = 0, \quad \text{for } j = 1, \dots, s.$$

This asserts that $\text{rank}((\partial H_j / \partial x_i)(x_{c+1}^0, \dots, x_n^0)) < s$. This is a contradiction. Q.E.D.

(4.16) Now we are ready to prove the positivity of g' . In this paragraph, we assume that f has an isolated singularity and is nondegenerate. Assume that Σ satisfies conditions (1)–(7).

(4.16.1) *Claim.* The function $\sum_{k=1}^n |g'_k(y)|^2$ is positive on the compact set $\pi_\Sigma^{-1}(0)$.

Proof. Assume that there is a point $y \in E_I^*$ such that $\sum_{k=1}^n |g'_k(y)|^2 = 0$. If $\prod_{j \in N_0 \cup N_-} y_j \neq 0$, then $g_k(y) = 0$, for $k = 1, \dots, n$, because of the definitions of g_k and g'_k . By (4.12), this contradicts nondegeneracy of f .

Assume that $\prod_{j \in N_0 \cup N_-} y_j = 0$. Lemmata (4.13)–(4.15) assert that nondegeneracy implies positivity of $\sum_{k \in M_I} |g'_k(y)|^2$. By (4.14),

$$\sum_{k \in M_I} |g'_k(y)|^2 = \sum_{k=1}^n |g'_k(y)|^2.$$

So this is a contradiction. Q.E.D.

(4.17) Now we are ready to complete the proof.

(4.17.1). First we write down the pullback of $|\text{grad } f|^2$ via π_σ :

$$\begin{aligned} |\text{grad } f|^{2(a(\sigma)y_\sigma)} &= \sum_{k=1}^n \left| \frac{\partial f}{\partial x_k}(a(\sigma)y_\sigma) \right|^2 \\ &= \sum_{k=1}^n \prod_{j \in N_+} |y_{\sigma,j}|^{2(l(a^j(\sigma)) - a_k^j(\sigma))} \cdot |g'_k(y_\sigma)|^2 \\ &\geq \prod_{j \in N_+} |y_{\sigma,j}|^{2(l(a^j(\sigma)) - m(a^j(\sigma)))} \sum_{k=1}^n |g'_k(y_\sigma)|^2. \end{aligned}$$

(4.17.2) Next we write down that of $|x|^2$:

$$\begin{aligned} |x|^{2(a(\sigma))} y_\sigma &= \sum_{k=1}^n |y_{\sigma,1}^{a_1^1(\sigma)} \cdots y_{\sigma,n}^{a_n^n(\sigma)}|^2 \\ &= \prod_{j \in N_+} |y_{\sigma,j}|^{2m(a^j(\sigma))} \sum_{k=1}^n |y_{\sigma,1}^{a_1^1(\sigma)-m(a^1(\sigma))} \cdots y_{\sigma,n}^{a_n^n(\sigma)-m(a^n(\sigma))}|^2. \end{aligned}$$

By (4.16), (4.17), and (4.9), we finally obtain $\alpha_0(f) \leq \max\{l(a^j(\sigma))/m(a^j(\sigma)), \text{ for } \sigma, j \text{ with } m(a^j(\sigma)) > 0\} - 1$, where $a(\sigma) = (a^1(\sigma), \dots, a^n(\sigma))$ for n -simplex σ of Σ . This completes the proof of (4.8).

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