SUBHARMONICITY, THE DISTANCE FUNCTION, AND a -ADMISSIBLE SETS

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ABSTRACT. Let Ω be an open subset of $\mathbb{R}^n$. The main result of this paper shows that a one-sided control on the curvature of $\partial\Omega$ is equivalent to subharmonicity of a function related to $\text{dist}(X, \partial\Omega)$. An application is given.

1. RESULTS

Throughout this paper Ω denotes a nonempty proper open subset of Euclidean space $\mathbb{R}^n$ ($n \geq 2$). Let $\overline{\Omega}$ and $\partial\Omega$ denote the closure and boundary of Ω respectively, and define

$$u(X) = \begin{cases} \text{dist}(X, \partial\Omega) & (X \in \Omega), \\ -\text{dist}(X, \partial\Omega) & (X \in \mathbb{R}^n \setminus \Omega). \end{cases}$$

The following result is due to Armitage and Kuran [1]. (Only connected sets Ω were considered in [1], but the proof does not require connectedness.)

Theorem A. *The set Ω is convex if and only if u is superharmonic in $\mathbb{R}^n$.*

In this paper we show that an analogous result holds when convexity is replaced by one-sided control on the curvature of $\partial\Omega$. Let $B(X, r)$ denote the open ball of center X and radius $r > 0$. An *inner maximal ball* (of radius r) is a ball $B(X, r) \subseteq \Omega$ such that $r = u(X)$. Such a ball *touches* $\partial\Omega$ in the sense that $\partial B(X, r) \cap \partial\Omega$ is nonempty. Let $a > 0$. Following Kuran [2] we say that Ω is *a -admissible* if every inner maximal ball of radius $r < a$ is contained in an inner maximal ball of radius a . If Ω is a -admissible, then every point of $\partial\Omega$ is touched by an inner maximal ball of radius a , but not conversely (see [2, §5]).

Let $\phi_n: (0, \infty) \rightarrow \mathbb{R}$ be given by $\phi_n(t) = t^{2-n}$ if $n \geq 3$, and by $\phi_2(t) = \log(1/t)$. Our main result is stated below.

Theorem 1. *The set Ω is a -admissible if and only if the function $X \mapsto \phi_n(a - u(X))$ is subharmonic on the set $\{X : u(X) < a\}$.*

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As an application of the “only if” part of Theorem 1 we obtain a short proof of the following, which is the main result in [2].

Corollary. *Let Ω be bounded and a -admissible, and let v be a positive superharmonic function on Ω . Then there is a positive constant k such that $v \geq ku$ on Ω .*

To see this, we note that v has a positive lower bound, K say, on the compact set $\{X : u(X) \geq a/2\}$. By the minimum principle we have

$$v(X) \geq K \frac{\phi_n(a - u(X)) - \phi_n(a)}{\phi_n(a/2) - \phi_n(a)} \geq K \frac{\max\{1, n-2\}a^{1-n}}{\phi_n(a/2) - \phi_n(a)} u(X)$$

on the set $\{X : 0 < u(X) < a/2\}$. Since u is bounded above, by L say, on Ω , we have $v \geq (K/L)u$ on $\{X : u(X) \geq a/2\}$. Thus the corollary is proved.

It is shown in [2, §5] that the corollary fails if the hypothesis of a -admissibility is weakened slightly to the condition that every $Z \in \partial\Omega$ is touched by an inner maximal ball of radius a .

2. PROOF OF THEOREM 1

2.1. We begin with the “only if” part of the theorem. For each $Y \in \mathbb{R}^n$ define $s_Y(X) = a - |X - Y|$. Let Ω be a -admissible and let $E = \{Y : B(Y, a) \subseteq \Omega\}$. It is clear that $u \geq \sup_{Y \in E} s_Y$ on $\mathbb{R}^n$. To see that equality holds on the set $\{X : u(X) < a\}$, we consider the following two cases.

(i) If $u(X) \leq 0$, there is a point $Z \in \partial\Omega$ such that $u(X) = -|X - Z|$. As observed in §1, there is a ball $B(P, a) \subseteq \Omega$ which touches $\partial\Omega$ at Z . Hence $u(X) = s_P(X)$.

(ii) If $0 < u(X) < a$, the inner maximal ball $B(X, u(X))$ is contained in another inner maximal ball $B(P, a)$. Hence, again, $u(X) = s_P(X)$.

Since the function $X \mapsto \phi_n(a - s_Y(X))$ is harmonic on $\mathbb{R}^n \setminus \{Y\}$, it follows that the continuous function $\phi_n(a - u(\cdot)) = \sup_{Y \in E} \phi_n(a - s_Y(\cdot))$ is subharmonic on $\{X : u(X) < a\}$.

2.2. The “if” part of the proof uses ideas from Armitage and Kuran [1] and Parker [3]. Let $\mathcal{M}(f; X, r)$ denote the mean of a function f over $\partial B(X, r)$. We begin by recording two lemmas.

Lemma A. *Let Y, Z_1, Z_2 be distinct points in $\mathbb{R}^n$ such that $|Y - Z_1| = |Y - Z_2| = d < a$, and let $v(X) = \min\{|X - Z_1|, |X - Z_2|\}$. Then there exists $r_0 \in (0, a - d)$ such that $\mathcal{M}(\phi_n(a - v(\cdot)); Y, r) < \phi_n(a - v(Y))$ for all $r \in (0, r_0)$.*

Lemma B. *Let $B(X, r) \subset \Omega$ be such that $\partial B(X, r) \cap \partial\Omega = \{Z\}$ and let $Q \in \overline{B(X, r)} \setminus \{Z\}$. Then there exists $\varepsilon_0 > 0$ such that $\overline{B(X + \varepsilon(Q - Z), r)} \subset \Omega$ for all $\varepsilon \in (0, \varepsilon_0)$.*

Lemma A is a modified version of [1, Lemma 2] and can be proved by similar reasoning. A sketch proof of Lemma B can be found in [3, p. 504].

Now let Ω be such that the function $\phi_n(a - u(\cdot))$ is subharmonic on the set $\{X : u(X) < a\}$. Let $0 < r < a$, let $B(X_0, r)$ be an inner maximal ball,

and choose $Z_0 \in \partial B(X_0, r) \cap \partial \Omega$. We are going to show that there is an inner maximal ball $B(Y_0, a)$ containing $B(X_0, r)$.

Let $Y \in B(Z_0, a) \cap \Omega$ and suppose that there are distinct points $Z_1, Z_2 \in \partial \Omega$ such that $|Y - Z_1| = |Y - Z_2| = u(Y)$. Let v be as in Lemma A. Since $u(Y) \leq |Y - Z_0| < a$, we can choose r_0 as in Lemma A. For $0 < \rho < r_0$ we have

$\mathcal{M}(\phi_n(a - u(\cdot)); Y, \rho) \leq \mathcal{M}(\phi_n(a - v(\cdot)); Y, \rho) < \phi_n(a - v(Y)) = \phi_n(a - u(Y))$, contradicting the fact that $\phi_n(a - u(\cdot))$ is subharmonic on a neighborhood of Y . Hence every point in $B(Z_0, a) \cap \Omega$ has a unique closest point in $\partial \Omega$.

Let $X_\varepsilon = X_0 + \varepsilon(X_0 - Z_0)$ and $B_\varepsilon = B(X_\varepsilon, r)$ for $\varepsilon > 0$. By Lemma B there is an $\varepsilon_0 > 0$ such that $\overline{B_\varepsilon} \subseteq \Omega$ for all $\varepsilon \in (0, \varepsilon_0)$. Now fix $\varepsilon \in (0, \varepsilon_0)$ and let

$$r_\varepsilon = \sup\{\rho \in [r, a] : \text{there exists } Y \text{ for which } B_\varepsilon \subseteq B(Y, \rho) \subseteq \Omega\}.$$

Clearly $r < r_\varepsilon \leq a$. There is a sequence of balls $B(P_m, \rho_m)$ such that $\rho_m \rightarrow r_\varepsilon$ and $B_\varepsilon \subseteq B(P_m, \rho_m) \subseteq \Omega$ for each m . By choosing a convergent subsequence of (P_m) , with limit P_0 say, it follows that $B_\varepsilon \subseteq B(P_0, r_\varepsilon) \subseteq \Omega$.

Now suppose that $r_\varepsilon < a - \varepsilon r$. Since

$$|P_0 - Z_0| \leq |P_0 - X_\varepsilon| + |X_\varepsilon - Z_0| \leq r_\varepsilon - r + (1 + \varepsilon)r < a,$$

there is a unique point Q_0 of $\partial \Omega$ that is closest to P_0 . By the maximality of r_ε we have $|P_0 - Q_0| = r_\varepsilon$, and so $\partial B(P_0, r_\varepsilon) \cap \partial \Omega = \{Q_0\}$. Let $Q \in \partial B_\varepsilon$ be chosen so that $|Q - P_0|$ is maximized. Clearly $Q \neq Q_0$, since $\overline{B_\varepsilon} \subseteq \Omega$. By Lemma B there is a $\delta_0 > 0$ such that

$$(1) \quad \overline{B(P_0 + \delta(Q - Q_0), r_\varepsilon)} \subseteq \Omega, \quad (0 < \delta < \delta_0).$$

For small $\delta > 0$ we also have

$$(2) \quad B_\varepsilon \subseteq B(P_0 + \delta(Q - Q_0), r_\varepsilon).$$

But (1) and (2) together contradict the maximality of r_ε . Hence $a - \varepsilon r \leq r_\varepsilon \leq a$.

Thus, for each large $m \in \mathbb{N}$, there is a ball $B(Y_m, r_m)$ such that $a - r/m \leq r_m \leq a$ and

$$B_{1/m} \subseteq B(Y_m, r_m) \subseteq \Omega.$$

It follows that there exists a point Y_0 such that

$$B(X_0, r) \subseteq B(Y_0, a) \subseteq \Omega,$$

as required.

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