SEPARATION AND VON NEUMANN INTERSECTION THEOREMS

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(Communicated by William J. Davis)

ABSTRACT. We give some separation theorems to extend the intersection theorem of von Neumann, Fan, and others, omitting hypotheses of convexity and local convexity for one of the coordinate spaces.

Let X, Y be two nonempty compact convex sets, each in a Euclidean space. Let E, F be closed subsets of $X \times Y$. Von Neumann's intersection theorem [6] asserts that, if for each $x \in X$ and $y \in Y$, both $E(y) = \{x' \in X | (x', y) \in E\}$ and $F(x) = \{y' \in Y | (x, y') \in F\}$ are nonempty and convex, then $E \cap F \neq \emptyset$. For the case where $Y = X$ and $F = \{(x, x) | x \in X\}$, this theorem reduces to Kakutani's fixed-point theorem [4]. Ky Fan extended the theorem to locally convex spaces in 1952 [1]. As a further extension, we prove the theorem allowing X or Y to be nonconvex in a nonlocally convex topological vector space.

Our main result is Theorem 3, a generalization of von Neumann's intersection theorem [6]. Its proof relies on some conclusions about separating disjoint graphs, which are stated below as Theorems 1 and 2. These elementary theorems may find additional uses of interest. Theorem 4 is a generalization of von Neumann's minimax theorem [5].

Let X, Y be topological spaces. A *correspondence* $U: X \rightarrow Y$ is a function from X to the family of subsets of Y . A correspondence $U: X \rightarrow Y$ is said to be with *open graph* (*closed graph*) if the set $\text{Gr } U = \{(x, y) \in X \times Y | y \in U(x)\}$ is open (closed) in $X \times Y$. We identify U and $\text{Gr } U$. Let $U^{-1}(y) = \{x \in X | y \in U(x)\}$. Let $\text{co } S$ denote the convex hull of the set S and $\text{Conv } S$ denote the convex closure of the set S where S is a subset of a topological vector space, conventionally Hausdorff. The empty set is both convex and compact.

The following is a separation theorem, which is a stronger form of Hausdorff separation principle.

Theorem 1. *Let X be a compact subset of a topological vector space E_1 and Y a compact subset of a locally convex space E_2 . Let U and D be two correspondences from X into Y of closed graphs such that for each $x \in X$,*

Received by the editors February 3, 1989 and, in revised form, June 6, 1989.
 1980 *Mathematics Subject Classification* (1985 Revision). Primary 54H25.

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 0002-9939/91 \$1.00 + \$.25 per page

$\text{Conv } U(x) \cap \text{Conv } D(x) = \emptyset$ and either $\text{Conv } U(x)$ or $\text{Conv } D(x)$ is compact. Then there exist two correspondences G' and G'' from E_1 into E_2 of open graphs such that

- (1) $U \subset G'$ and $D \subset G''$;
- (2) $G' \cap G'' = \emptyset$; and
- (3) $G'(x)$ and $G''(x)$ are convex for all $x \in E_1$.

Proof. For each $x \in X$ there exists a neighborhood W_x of 0 in E_2 such that

(a) $[\text{Conv } U(x) + W_x] \cap [\text{Conv } D(x) + W_x] = \emptyset$. We adopt the convention that $\emptyset + W_x = \emptyset$. Let V'_x be a convex open neighborhood of 0 in E_2 such that

(b) $V'_x + V'_x \subset W_x$. Since U and D are closed and hence upper semicontinuous, there exists an open, balanced neighborhood V_x of 0 in E_1 such that $U(x + V_x) \subset U(x) + V'_x$ and $D(x + V_x) \subset D(x) + V'_x$. Let V''_x be an open, balanced neighborhood of 0 in E_1 such that $v''_x + V''_x + V''_x \subset V_x$. By compactness there exist finitely many points $x_1, \dots, x_n \in X$ such that $X \subset \bigcup_{i=1}^n (x_i + V''_{x_i})$. Set $V = \bigcap_{i=1}^n V''_{x_i}$, $V' = \bigcap_{i=1}^n V'_{x_i}$.

Since $G_1 = U + (V \times V')$ and $G_2 = D + (V \times V')$ are open correspondences from E_1 into E_2 , so are

$$G' = \{(x, y) : x \in E_1 \text{ and } y \in \text{co } G_1(x)\},$$

$$G'' = \{(x, y) : x \in E_1 \text{ and } y \in \text{co } G_2(x)\}.$$

Moreover, G' and G'' clearly have convex values and contain U and D , respectively. Now suppose for some $x \in E_1$ there exist $(x, y') \in G'$ and $(x, y'') \in G''$. Then

$$y' \in \text{co } G_1(x) = \text{co} \left(\bigcup_{x' \in x+V} [U(x') + V'] \right) = \text{co}(U(x+V) + V').$$

Thus $U(x+V)$ is nonempty and there exists $x' \in (x+V) \cap X$. Fix j , $1 \leq j \leq n$, such that $x' \in x_j + V''_{x_j}$, and note that $x+V \subset (x'+V) + V \subset x_j + V''_{x_j} + V + V \subset x_j + V_{x_j}$. It follows that

$$\begin{aligned} y' &\in \text{co}(U(x_j + V_{x_j}) + V') \subset \text{co}(U(x_j) + V'_{x_j} + V') \\ &\subset [\text{co } U(x_j)] + V'_{x_j} + V'_{x_j}, \end{aligned}$$

since the sum of convex sets is convex. Symmetrically, $y'' \in [\text{co } D(x_j)] + V'_{x_j} + V'_{x_j}$. Therefore y' and y'' are distinct, belonging to disjoint sets by (b) and (a). Consequently, G' and G'' are disjoint, and the proof is complete.

The following open correspondence separation theorem yields a single-valued continuous selection function and allows one of the original correspondences to have nonconvex values.

Theorem 2. *Let X be a compact subset of a topological vector space E_1 and Y a compact subset of a locally convex space E_2 . Let the correspondences U and D from X into Y with closed graphs be such that*

$$U \cap D = \emptyset \quad \text{and} \quad U(x) \text{ is convex for all } x \in X.$$

Then there exists a correspondence G from E_1 into E_2 of open graph such that

- (1) $U \subset G$;
- (2) $G \cap D = \emptyset$; and
- (3) $G(x)$ is convex for all $x \in X$.

Furthermore, there exist a finite subset S of Y and a (single-valued) continuous function $f: X' \rightarrow \text{co}S$ such that $f(x) \in G(x)$ for each $x \in X'$, where $X' = \{x \in X \mid U(x) \neq \emptyset\}$.

Proof. By Theorem 1, for each $(x, y) \in D$ there exist two open sets $G'_{(x, y)}$ and $G''_{(x, y)}$ in $X \times Y$ such that

- (i) $U \subset G'_{(x, y)}$ and $\{(x, y)\} \subset G''_{(x, y)}$;
- (ii) $G'_{(x, y)} \cap G''_{(x, y)} = \emptyset$; and
- (iii) $G'_{(x, y)}(x')$ is convex for all $x' \in X$.

Then $D \subset \bigcup_{(x, y) \in D} G''_{(x, y)}$. Since D is compact, there exists a finite subcover $\{G''_{(x_i, y_i)}\}_{i=1}^k$ of D . Let $G = \bigcap_{i=1}^k G'_{(x_i, y_i)}$. Then $G \cap G''_{(x_i, y_i)} = \emptyset$ for all $i = 1, \dots, k$ and hence $G \cap D = \emptyset$. By the properties of $G'_{(x_i, y_i)}$, $U \subset G$, G is open in $X \times Y$, and $G(x)$ is convex for all $x \in X$.

Since G is open and U is compact, there is a finite set $S \subset Y$ such that $G(x) \cap S \neq \emptyset$ for each $x \in X'$. Define $H: X' \rightarrow \text{co}S$ by $H(x) = G(x) \cap \text{co}S$. Then H is with open graph and convex nonempty values. By Theorem 3.1 in [7], there is a continuous function $f: X' \rightarrow \text{co}S$ such that $f(x) \in H(x)$ for all $x \in X'$. This completes the proof.

Theorem 3. *Let X be a compact subset of a topological vector space E_1 and Y a convex compact subset of a locally convex space E_2 . Let U and D be two correspondences with closed graphs from X into Y such that $U(x)$ is nonempty and convex for all $x \in X$ and $D^{-1}(y)$ is nonempty and convex for all $y \in Y$. Then $U \cap D \neq \emptyset$.*

Proof. Suppose that $U \cap D = \emptyset$. By Theorem 2 there is a correspondence G from E_1 into E_2 with open graph such that

- (1) $U \subset G$;
- (2) $G \cap D = \emptyset$; and
- (3) $G(x)$ is convex for all $x \in X$.

Furthermore, there exist a finite subset S of Y and a (single-valued) continuous function $f: X \rightarrow \text{co}S$ such that $f(x) \in G(x)$ for each $x \in X$. Define $q: \text{co}S \rightarrow X$ by $q(y) = D^{-1}(y)$ for all $y \in \text{co}S$. Then by relaxing n -simplex

to $\text{co} S$ in Lemma 2 of [3], there exists $z \in \text{co} S$ such that $z \in f(q(z))$. This implies that $G \cap D \neq \emptyset$, which is a contradiction, and hence $U \cap D \neq \emptyset$.

Using Theorem 3, one can easily prove:

Theorem 4. *Let X be a compact subset of a topological vector space E_1 and Y a compact convex subset of a locally convex space E_2 . Let f be a real-valued continuous function on $X \times Y$. If, for each $x_0 \in X$, $y_0 \in Y$, the sets:*

$$\left\{ x \in X \mid f(x, y_0) = \max_{\zeta \in X} f(\zeta, y_0) \right\}$$

and

$$\left\{ y \in Y \mid f(x_0, y) = \min_{\eta \in Y} f(x_0, \eta) \right\}$$

are convex, then $\max_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \max_{x \in X} f(x, y)$.

ACKNOWLEDGMENT

The author would like to thank the referee for suggestions that lead to a better presentation of this paper and a much simpler and shorter Theorem 1 that does not use the Hahn–Banach Theorem.

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