RELATING SPANS OF SOME CONTINUA HOMEOMORPHIC TO S^n

THELMA WEST

(Communicated by James E. West)

ABSTRACT. It has been conjectured that $\sigma^*(X) \geq \frac{1}{2}\sigma_0^*(X)$ for each nonempty connected metric space X . In this paper we show that $\sigma^*(X) \geq (\sqrt{3}/2)\sigma_0^*(X)$ when $X \subset R^n$ is homeomorphic to S^{n-1} for $n = 2, 3, \dots$ and A is convex where A is the bounded component of $R^n - X$. We also show that under certain conditions a lower bound for the ratio $\sigma^*(X)/\sigma_0^*(X)$ is larger than $\sqrt{3}/2$. It has also been conjectured that $\sigma^*(X) \geq \sigma(X)/2$ and that $\sigma_0^*(X) \geq \sigma_0(X)/2$ for each nonempty connected metric space X . We show that these two inequalities hold when $X \subset R^n$ is homeomorphic to S^{n-1} for $n = 3, 4, \dots$ and A is convex where A is the bounded component of $R^n - X$.

Generally speaking, the spans of an object are connectedness type analogues of its diameter. We follow the definitions from [1]. Let X be a nonempty connected metric space. The standard projections of the product $X \times X$ onto X are denoted by p_1 and p_2 , that is, $p_1(x, x') = x$ and $p_2(x, x') = x'$ for $(x, x') \in X \times X$. The *surjective span* $\sigma^*(X)$ of X is the least upper bound of real numbers α such that there exist nonempty connected sets $C_\alpha \subset X \times X$ with $\text{dist}(x, x') \geq \alpha$ for $(x, x') \in C_\alpha$ and $p_1(C_\alpha) = p_2(C_\alpha) = X$. Relaxing the last condition to $p_1(C_\alpha) = p_2(C_\alpha)$, $p_2(C_\alpha) = X$, or $p_1(C_\alpha) \subset p_2(C_\alpha)$, one obtains the definitions of the *span* $\sigma(X)$, the *surjective semispan* $\sigma_0^*(X)$, and the *semispan* $\sigma_0(X)$ of X , respectively. Hence,

$$(1) \quad 0 \leq \sigma^*(X) \leq \sigma(X) \leq \sigma_0(X) \leq \text{diam } X,$$

$$(2) \quad 0 \leq \sigma^*(X) \leq \sigma_0^*(X) \leq \sigma_0(X) \leq \text{diam } X.$$

From the definitions it is clear that if T is an isometry, then $\sigma(X) = \sigma[T(X)]$, $\sigma^*(X) = \sigma^*[T(X)]$, $\sigma_0(X) = \sigma_0[T(X)]$, and $\sigma_0^*(X) = \sigma_0^*[T(X)]$. When X is a continuum, that is a nonempty, compact, connected, metric space, we need only consider those sets C_α that are closed in $X \times X$. It has been shown in the above definitions that when X is a continuum, the requirement $\text{dist}(x, x') \geq \alpha$ for $(x, x') \in C_\alpha$ can be replaced by $\text{dist}(x, x') = \alpha$ for $(x, x') \in C_\alpha$ (see [2, p. 169]).

Received by the editors October 14, 1989.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 54F15; Secondary 54F20.

Key words and phrases. Span, convex continua.

There is a conjecture that $\sigma^*(X) \geq \sigma_0^*(X)/2$ for each nonempty connected metric space X (see [1, p. 38]). In this paper we show that $\sigma^*(X) \geq (\sqrt{3})/2\sigma_0^*(X)$ when $X \subset R^n$ is homeomorphic to S^{n-1} for $n = 2, 3, \dots$ and A is convex where A is the bounded component of $R^n - X$. We also show that under certain conditions a lower bound for the ratio $\sigma^*(X)/\sigma_0^*(X)$ is larger than $\sqrt{3}/2$. It has also been conjectured that $\sigma^*(X) \geq \sigma(X)/2$ and $\sigma_0^*(X) \geq \sigma_0(X)/2$ for each nonempty connected metric space X (see [1, p. 37]). We show that these two inequalities hold when $X \subset R^n$ is homeomorphic to S^{n-1} for $n = 3, 4, \dots$ and A is convex where A is the bounded component of $R^n - X$. In the case when X is homeomorphic to S^1 an easily established fact is that $\sigma(X) = \sigma^*(X)$ and $\sigma_0^*(X) = \sigma_0(X)$.

We will use ρ to denote the usual metrics on R^2 , R^n , and $R^n \times R^n$. It will be clear in all cases which metric ρ represents. Let $a, b \in R^n$. We denote the straight line segment in R^n with endpoints a and b by $\overline{ab}$. We use the symbol o to denote the origin in R^n .

These facts we will use in this paper:

- (3) If $X \subset R^n$ and T is an isometry from X into R^n , then for $\overline{xy} \subset X$, $T[\overline{xy}] = \overline{T(x)T(y)}$.

This is trivial and we exclude the proof.

- (4) Let $p_t \in \{(x, y) | (x - d)^2 + y^2 = t^2 d^2, \text{ where } 0 \leq t \leq 1\} \cap \{(x, y) | x^2 + y^2 \leq d^2\}$. Then $\rho((d, 0), \overline{op_t}) \geq dt\sqrt{4 - t^2}/2$ (see [4]).

Lemma 1. If $A \subset R^n$ where $n = 1, 2, \dots$ is a convex set, then $\overline{A}$ is also convex.

Lemma 2. Let $X \subset R^n$ where $n = 2, 3, \dots$ be homeomorphic to S^{n-1} and A be the bounded component of $R^n - X$. If r is any ray such that $r \cap A \neq \emptyset$, then $r \cap X \neq \emptyset$.

Lemma 3. If $T: R^n \rightarrow R^n$ where $n = 1, 2, \dots$ is an isometry and A is a convex set in R^n , and $T[A]$ is also convex.

Lemma 4. Let $X \subset R^n$ where $n = 2, 3, \dots$ be homeomorphic to S^{n-1} . Assume that A is convex where A is the bounded component of $R^n - X$. Let x and y be distinct points of X . Let $z \in R^n - \overline{xy}$. The function f where $f: \overline{xy} \rightarrow X$ and $f(t)$ is defined to be the point of X on $\overline{zt}$ farthest from z , is continuous.

Proof. The set $\overline{xy} \subset A \cup X$ since $\overline{A} = A \cup X$ and $\overline{A}$ is convex by Lemma 1. If $t \in \overline{xy}$, then either $t \in A$ and $(\overline{zt} - \overline{zt}) \cap X \neq \emptyset$ by Lemma 2 or $t \in X$. So for each $t \in \overline{xy}$, there is a point $x' \in \overline{zt} \cap X$ such that $\rho(z, x') \geq \rho(z, t)$. Also, for each $t \in \overline{xy}$, there is a point of X on $\overline{zt}$ that is farthest from z , because X is compact. Hence, f is well defined.

If $z \in \overrightarrow{xy} - \overline{xy}$, then f is a constant function. Hence, it is continuous. We just need to examine the case where $z \in R^n - \overrightarrow{xy}$.

Assume that $z \in R^n - \overleftarrow{xy}$. If f is not continuous at a point $t \in \overline{xy}$, then there is a sequence of distinct points in $\overline{xy}$, $\{t_i\}_{i=1}^\infty$, that converges to t , but the sequence $\{f(t_i)\}_{i=1}^\infty$ does not converge to $f(t)$. We know that a subsequence of $\{f(t_i)\}_{i=1}^\infty$ converges because $\{f(t_i)\}_{i=1}^\infty \subset X$ and X is compact. We will assume that $\{f(t_i)\}_{i=1}^\infty$ itself converges to $x'' \in X$. Clearly, $x'' \in \overrightarrow{zt}$. Also, $x'' \in \overline{zf(t)} - \{f(t)\}$ since $f(t)$ is defined to be the point of X on $\overrightarrow{zt}$ that is farthest from z . For each $i \in N$, $\rho(f(t_i), z) \geq \rho(t_i, z)$, hence $\rho(x'', z) \geq \rho(t, z)$ and $\rho(f(t), z) > \rho(x'', z) \geq \rho(t, z)$.

An infinite number of elements of the sequence $\{t_i\}_{i=1}^\infty$ is either on $\overline{xt}$ or $\overline{ty}$. We can assume that the entire sequence $\{t_i\}_{i=1}^\infty$ is contained in either $\overline{xt}$ or $\overline{ty}$.

If $\{t_i\}_{i=1}^\infty \subset \overline{xt}$ then $x \neq t$. The line segment $\overline{f(x)f(t)}$ does not contain x'' , because $f(x) \notin \overrightarrow{zt}$, $f(t) \in \overrightarrow{zt}$, and $x'' \in \overrightarrow{zt}$. Hence, there is an $\varepsilon > 0$ such that $B(x'', \varepsilon) \cap \overline{f(x)f(t)} = \emptyset$.

For each $i \in N$, let p_i be the point such that $\{p_i\} = \overrightarrow{zt_i} \cap \overline{f(x)f(t)}$. So, $f(t_i) \in (\overrightarrow{zt_i} - \overline{zp_i}) \cup \{p_i\}$, $\overline{f(t_i)x''} \cap \overline{f(x)f(t)} \neq \emptyset$, and $\rho(f(t_i), x'') \geq \rho(x'', \overline{f(x)f(t)}) > \varepsilon$. Hence $\{f(t_i)\}_{i=1}^\infty$ cannot converge to x'' .

In the case where $\{t_i\}_{i=1}^\infty \subset \overline{ty}$, the proof that $\{f(t_i)\}_{i=1}^\infty$ cannot converge to x'' is similar. Consequently, f is a continuous function.

The following lemma follows easily from Lemma 1.1 of [3, p. 6].

Lemma 5. Let $X \subset R^n$ where $n = 2, 3, \dots$ be homeomorphic to S^{n-1} . Assume that A is convex where A is the bounded component of $R^n - X$. Let x and y be distinct points of X . If $(\overline{xy} - \{x, y\}) \cap X \neq \emptyset$ then $\overline{xy} \subset X$.

Theorem 1. Let $X \subset R^n$ where $n = 2, 3, \dots$ be homeomorphic to S^{n-1} . Assume that A is convex where A is the bounded component of $R^n - X$. Then

$$\sigma^*(X) \geq [\sqrt{4 - (\sigma_0^*(X)/\text{diam } X)^2}/2]\sigma_0^*(X).$$

Proof. Let $C \subset X \times X$ such that C is closed and connected, $p_2(C) = X$ and $\rho(x, y) = \sigma_0^*(X)$ for each $(x, y) \in C$.

Let $a, b \in X$ such that $\rho(a, b) = \text{diam } X = d$. There must be points $a_1, b_1 \in X$ such that (a_1, b) and (b_1, a) are elements of C since $p_2(C) = X$.

We define a set G in the following manner. Case 1. $a_1 \in \overline{ab}$. In this case we let $G = \overline{aa_1}$. By Lemma 5, $G \subset X$. The set G is closed and connected and $\rho(G, b) = \rho(a_1, b) = \sigma_0^*(X)$.

Case 2. $a_1 \notin \overline{ab}$. In this case let p be the point on $\overline{ab}$ which is closest to a_1 . The point $p \neq a$ since $\rho(a, b) = d$ and $X \subseteq \overline{B(a, d)} \cap \overline{B(b, d)}$, where $\overline{B}(x, d) = \{y \in R^n | \rho(x, y) \leq d\}$. We define a function $g: \overline{aa_1} \rightarrow X$ where $g(t)$ is the point of X on the ray $\overrightarrow{pt}$ farthest from p . The point p is an

element of $R^n - \overline{aa_1}$, so g is well defined and continuous by Lemma 4. Hence $g[\overline{aa_1}]$ is compact and connected.

In this case we let $G = g[\overline{aa_1}] \cup \overline{a_1g(a_1)}$. If $g(a_1) = a_1$ then $\overline{a_1g(a_1)} = \{a_1\} \subset X$. If $g(a_1) \neq a_1$ then $\overline{a_1g(a_1)} \subset X$, because $a_1, g(a_1) \in X$ and either $p \in X$ and $\overline{pg(a_1)} \subset X$ by Lemma 5 or $z \in (\overline{a_1p} - \overline{a_1p}) \cap X \neq \emptyset$ by Lemma 2 and $\overline{zg(a_1)} \subset X$ by Lemma 5. Hence, $G \subset X$ and G is connected because $g[\overline{aa_1}]$ and $\overline{a_1g(a_1)}$ are both connected and $g(a_1) \in g[\overline{aa_1}] \cap \overline{a_1g(a_1)}$. Also, $\rho(G, b) \geq \rho(\overline{aa_1}, b)$.

Next we will determine a lower bound for $\rho(\overline{aa_1}, b)$. Let T be an isometry from X into R^n such that $T(a) = o$, $T(b) = (d, 0, \dots, 0)$, and $T(a_1) = q$ where q is a point in the plane $R \times R \times \{0\} \times \dots \times \{0\}$. Hence, $T[\overline{aa_1}] = \overline{oq}$ and $\rho(a_1, b) = \rho((d, 0, \dots, 0), q) = \sigma_0^*(X)$. Let $\sigma_0^*(X) = dt$ where $0 < t \leq 1$. Note that $T[X] \subset \{(x_1, \dots, x_n) | x_1^2 + \dots + x_n^2 \leq d^2\} \cap \{(x_1, \dots, x_n) | (x_1 - d)^2 + x_2^2 + \dots + x_n^2 \leq d^2\}$. Consequently, $q \in \{(x_1, x_2, 0, \dots, 0) | (x_1 - d)^2 + x_2^2 = t^2 d^2\} \cap \{(x_1, x_2, 0, \dots, 0) | x_1^2 + x_2^2 \leq d^2\}$. Hence, $\rho(\overline{aa_1}, b) \geq \rho(\overline{oq}, (d, 0, \dots, 0)) \geq dt\sqrt{4 - t^2}/2$ by (4).

In both of the above cases, G is closed and connected and $\rho(G, b) \geq dt\sqrt{4 - t^2}/2$. Also, note that $g(a) = a$ since $\rho(a, b) = \text{diam } X$ and $p \in \overline{ab}$. Hence, in both cases $a \in G$.

We define a set H is a similar way.

Case 1. $b_1 \in \overline{ab}$. In this case let $H = \overline{b_1b}$. By Lemma 5, $H \subset X$. The set H is closed and connected and $\rho(H, a) = \rho(a, b_1) = \sigma_0^*(X)$.

Case 2. $b_1 \notin \overline{ab}$. In this case let p' be the point on $\overline{ab}$ that is closest to b_1 . The point $p' \neq b$ since $\rho(a, b) = \text{diam } X = d$ and $X \subseteq \overline{B}(a, d) \cap \overline{B}(b, d)$. We define a function $h: \overline{b_1b} \rightarrow X$ where $h(t)$ is the point of X on the ray $\overline{p't}$ farthest from p' . The point p' is an element of $R^n - \overline{aa_1}$, so h is well defined and it is continuous by Lemma 4. Hence, $h[\overline{b_1b}]$ is compact and connected.

In this case we let $H = h[\overline{b_1b}] \cup \overline{b_1h(b_1)}$. By an argument similar to the one given for the set G in Case 2 above, H is closed and connected, $\rho(H, a) \geq \rho(\overline{b_1b}, a)$ and $\rho(\overline{b_1b}, a) \geq dt(\sqrt{4 - t^2}/2)$.

In both cases, H is closed and connected and $\rho(H, a) \geq dt(\sqrt{4 - t^2}/2)$. Also, note that $h(b) = b$ since $\rho(a, b) = \text{diam } X$ and $p' \in \overline{ab}$. Hence, in both cases $b \in H$.

Let $C' = C \cup (G \times \{b\}) \cup (\{a\} \times H) \cup C^{-1}$. So, $p_1(C') = p_2(C') = X$ since $X = p_2(C) \subseteq p_2(C')$ and $X = p_2(C) = p_1(C^{-1}) \subseteq p_1(C')$. Each of the sets, C , $G \times \{b\}$, $\{a\} \times H$, and C^{-1} is connected. Also, $(a_1, b) \in C \cap (G \times \{b\})$, $(a, b) \in (G \times \{b\}) \cap (\{a\} \times H)$, and $(a, b_1) \in (\{a\} \times H) \cap C^{-1}$. Hence, C' is connected.

Also, for $(x, y) \in C'$, $\rho(x, y) \geq dt(\sqrt{4 - t^2}/2)$. Hence, $\sigma^*(X) \geq dt(\sqrt{4 - t^2}/2) = \sigma_0^*(X)(\sqrt{4 - (\sigma_0^*(X)/\text{diam } X)^2}/2)$.

Corollary 1. Let $X \subset R^n$ where $n = 2, 3, \dots$ be homeomorphic to S^{n-1} . Assume that A is convex where A is the bounded component of $R^n - X$. Then $\sigma^*(X) \geq (\sqrt{3}/2)\sigma_0^*(X)$.

Theorem 2. Let $X \subset R^n$ where $n = 2, 3, \dots$ be homeomorphic to S^{n-1} . Assume that A is convex where A is the bounded component of $R^n - X$. Then $\sigma^*(X) \geq \sigma_0(X)/2$.

Proof. Let $C \subset X \times X$ be closed and connected, $p_1(C) \subset p_2(C)$ and $\rho(x, y) = \sigma_0(X)$ for each $(x, y) \in C$. Let $a, b \in p_2(C)$ such that $\rho(a, b) = \text{diam } p_2(C) = d$.

Let T be an isometry from R^n into R^n such that $T(a) = o$ and $T(b) = (d, 0, \dots, 0)$. Let $X_L = \{x \in X | \pi_1[T(x)] \leq d/2\}$ and $X_R = \{x \in X | \pi_1[T(x)] \geq d/2\}$ where π_1 is the standard projection from R^n into the first factor space.

Next, we will see that X_L and X_R are both connected. Suppose that X_L is not connected. Then there exist nonempty open sets in R^n , U_1 and U_2 such that $(U_1 \cap X_L) \cup (U_2 \cap X_L) = X_L$ and $(U_1 \cap X_L) \cap (U_2 \cap X_L) = \emptyset$. Let $x_1 \in U_1 \cap X_L$ and $x_2 \in U_2 \cap X_L$. Assume that $b \in \overrightarrow{x_1 x_2}$. Then $\overline{x_1 x_2} \subset X$ by Lemma 5 since b, x_1 and x_2 are elements of X . Let $t \in \overline{x_1 x_2}$. Then $T(t) \in \overline{T(x_1)T(x_2)}$ by (3) and $\pi_1(T(t)) \leq d/2$ because $\pi_1(T(x_1)) \leq d/2$ and $\pi_1(T(x_2)) \leq d/2$. So, $\overline{x_1 x_2} \subset X_L$ contrary to the assumption that x_1 and x_2 are in different components of X_L . Hence, $b \notin \overrightarrow{x_1 x_2}$.

We can define a function f where $f: \overline{x_1 x_2} \rightarrow X$ and $f(t)$ is defined to be the point of X on $\overrightarrow{bt}$ farthest from b . By Lemma 4 f is continuous. Hence, $f[\overline{x_1 x_2}]$ is compact and connected.

Let $B = \overline{x_1 f(x_1)} \cup \overline{f[\overline{x_1 x_2}] \cup \overline{x_2 f(x_2)}}$. The point $x_2 \in \overline{bf(x_2)}$ and $x_2, f(x_2)$, and b are elements of X so by Lemma 4, $\overline{x_2 f(x_2)} \subset X$. Similarly, $\overline{x_1 f(x_1)} \subset X$. Hence, $B \subset X$. Also, B is connected since $\overline{x_1 f(x_1)}$, $f[\overline{x_1 x_2}]$, and $\overline{x_2 f(x_2)}$ are each connected and $f(x_1) \in \overline{x_1 f(x_1)} \cap f[\overline{x_1 x_2}]$ and $f(x_2) \in f[\overline{x_1 x_2}] \cap \overline{x_2 f(x_2)}$.

Next, we will show that $B \subset X_L$. Let $z \in B$. There is a $t \in \overline{x_1 x_2}$ such that $z \in (\overrightarrow{bt} - \overrightarrow{bt}) \cup \{t\}$. The image of $\overrightarrow{bt}$ under T is $\overrightarrow{T(b)T(t)} = \overrightarrow{(d, 0, \dots, 0)T(t)}$ and $\pi_1(\overrightarrow{T(b)T(t)}) = \overrightarrow{do}$. The image of $\overrightarrow{bt}$ under T is $\overrightarrow{T(b)T(t)} = \overrightarrow{(d, 0, \dots, 0)T(t)}$. Hence, $T(z) \in [(\overrightarrow{(d, 0, \dots, 0)T(t)} - \overrightarrow{(d, 0, \dots, 0)T(t)}) \cup \{T(t)\}]$. Consequently, $\pi_1[T(z)] \in [\overrightarrow{do} - \overrightarrow{d\pi_1(T(t))}] \cup \{\pi_1[T(t)]\}$, $\pi_1[T(z)] \leq \pi_1[T(t)]$, and $\pi_1[T(t)] \leq \max\{\pi_1[T(x_1)], \pi_1[T(x_2)]\} \leq d/2$, so $z \in X_L$ and $B \subset X_L$. Hence, X_L is connected since B is connected. Similarly, X_R is connected. Note that if $x \in X_L$, then $\rho(x, b) \geq d/2$ and if $x \in X_R$, then $\rho(x, a) \geq d/2$.

There must be points $a_1, b_1 \in X$ such that (a_1, b) and (b_1, a) are elements of C since $a, b \in p_2(C)$.

We define a set G in the following manner.

Case 1. $a_1 \in \overline{ab}$. In this case we let $G = \overline{aa_1}$. By Lemma 5 $G \subset X$. The set G is closed and connected and $\rho(G, b) = \rho(a_1, b) = \sigma_0(X)$.

Case 2. $a_1 \notin \overline{ab}$. In this case let p be the point on $\overline{ab}$ closest to a_1 . The point $p \neq a$ because $\rho(a, b) = d$, $p_2(C) \subseteq \overline{B}(a, d) \cap \overline{B}(b, d)$ and $a_1 \in p_1(C) \subset p_2(C)$. We define a function $g: \overline{aa_1} \rightarrow X$ where $g(t)$ is the point of X on the ray $\overrightarrow{pt}$ farthest from p . The point p is an element of $R^n - \overline{aa_1}$, so g is well defined and is continuous by Lemma 4. Hence $g[\overline{aa_1}]$ is compact and connected.

In this case we let $G = \overline{ag(a)} \cup g[\overline{aa_1}] \cup \overline{a_1g(a_1)}$. If $g(a) = a$ then $\overline{ag(a)} = \{a\} \subset X$. If $g(a) \neq a$ then $\overline{ag(a)} \subset X$ by Lemma 5, since a , $g(a)$, and b are elements of X and $\overline{g(a)b}$. If $g(a_1) = a_1$ then $\overline{a_1g(a_1)} = \{a_1\} \subset X$. If $g(a_1) \neq a_1$ then $\overline{a_1g(a_1)} \subset X$, since a_1 , $g(a_1) \in X$ and either $p \in X$ and $\overline{pg(a_1)} \subset X$ by Lemma 5 or there exists a point z such that $z \in (\overline{a_1p} - \overline{a_1p}) \cap X$ by Lemma 2 and $\overline{zg(a_1)} \subset X$ by Lemma 5. So, $G \subset X$. Also, G is connected since $\overline{ag(a)}$, $g[\overline{aa_1}]$, and $\overline{a_1g(a_1)}$ are each connected and $g(a) \in \overline{ag(a)} \cap g[\overline{aa_1}]$ and $g(a_1) \in g[\overline{aa_1}] \cap \overline{a_1g(a_1)}$. Note that $\rho(G, b) \geq \rho(\overline{aa_1}, b)$.

Let T be an isometry from X into R^n such that $T(a) = o$, $T(b) = (d, 0, \dots, 0)$ and $T(a_1) = q$ where q is a point in the plane $R \times R \times \{0\} \times \dots \times \{0\}$. Hence, $T[\overline{aa_1}] = oq$ and $\rho(a_1, b) = \rho((d, 0, \dots, 0), q) = \sigma_0(X)$. Let $\sigma_0(X) = dt$ where $0 < t \leq 1$. Note that $T[p_2(C)] \subset \{(x_1, \dots, x_n) | x_1^2 + \dots + x_n^2 \leq d^2\} \cap \{(x_1, x_2, \dots, x_n) | (x_1 - d)^2 + x_2^2 + \dots + x_n^2 \leq d^2\}$, since $\text{diam } T[p_2(C)] = \text{diam } p_2(C) = d$, $a, b \in p_2(C)$, and $\rho(a, b) = d$. Consequently, $q \in \{(x_1, x_2, 0, \dots, 0) | (x_1 - d)^2 + x_2^2 = t^2 d^2\} \cap \{(x_1, x_2, 0, \dots, 0) | x_1^2 + x_2^2 \leq d^2\}$ since $T(a_1) = q$, $\rho(a_1, b) = \rho(q, (d, 0, \dots, 0)) = dt$, $a_1 \in p_1(C) \subset p_2(C)$, and $q \in T[p_2(C)]$. Hence, $\rho(\overline{aa_1}, b) \geq \rho(oq, (d, 0, \dots, 0)) \geq dt\sqrt{4 - t^2}/2$ by (4). In both of the above cases, G is closed and connected and $\rho(G, b) \geq dt(\sqrt{4 - t^2}/2)$.

We define a set H in a similar manner.

Case 1. $b_1 \in \overline{ab}$. In this case, let $H = \overline{b_1b}$. By Lemma 5 $H \subset X$. The set H is closed and connected and $\rho(H, a) = \rho(a, b_1) = \sigma_0(X)$.

Case 2. $b_1 \notin \overline{ab}$. In this case let p' be the point on $\overline{ab}$ closest to b_1 . The point $p' \neq b$ since $\rho(a, b) = \text{diam } p_2(C) = d$ and $p_2(C) \subseteq \overline{B}(a, d) \cap \overline{B}(b, d)$. We define a function $h: \overline{b_1b} \rightarrow X$ where $h(t)$ is the point of X on the ray $\overrightarrow{p't}$ farthest from p' . The point p' is an element of $R^n - \overline{b_1b}$, so h is well defined and it is continuous by Lemma 4. Hence, $h[\overline{b_1b}]$ is compact and connected.

In this case we let $H = \overline{bh(b)} \cup h[\overline{b_1b}] \cup \overline{b_1h(b_1)}$. By an argument similar to the one given for the set G in Case 2 above, H is closed and connected and $\rho(H, a) \geq \rho(\overline{b_1b}, a) \geq dt(\sqrt{4 - t^2}/2)$.

In both cases H is closed and connected and $\rho(H, a) \geq dt(\sqrt{4 - t^2}/2)$.

Let $C' = C \cup (G \times \{b\}) \cup (X_L \times \{b\}) \cup (\{a\} \times X_R) \cup (H \times \{a\}) \cup (X_R \times \{a\}) \cup (\{b\} \times X_L)$. The set $p_2(C') = X$ since $p_2(\{a\} \times X_R) = X_R$, $p_2(\{b\} \times X_L) = X_L$, and $X_R \cup X_L = X$. Similarly, $p_1(C') = X$.

Each of the sets C , $G \times \{b\}$, $X_L \times \{b\}$, $\{a\} \times X_R$, $H \times \{a\}$, $X_R \times \{a\}$, and $\{b\} \times X_L$ is connected. The set C' is connected since $(a_1, b) \in C \cap (G \times \{b\})$, $(a, b) \in (G \times \{b\}) \cap (X_L \times \{b\}) \cap (\{a\} \times X_R)$, $(b_1, a) \in C \cap (H \times \{a\})$, and $(b, a) \in (H \times \{a\}) \cap (X_R \times \{a\}) \cap (\{b\} \times X_L)$.

Next, we will see that if $(x, y) \in C'$, then $\rho(x, y) \geq \sigma_0(X)/2$. Let $(x, y) \in C'$. If $(x, y) \in C$, then $\rho(x, y) = \sigma_0(X)$. If $(x, y) \in (G \times \{b\}) \cup (H \times \{a\})$ then $\rho(x, y) \geq dt(\sqrt{4-t^2}/2) \geq (\sqrt{3}/2)\sigma_0(X)$. If $(x, y) \in (\{a\} \times X_R) \cup (X_R \times \{a\}) \cup (\{b\} \times X_L) \cup (X_L \times \{b\})$ then $\rho(x, y) \geq d/2 \geq \sigma_0(X)/2$. Consequently, $\sigma^*(X) \geq \sigma_0(X)/2$.

Corollary 2. Let $X \subset R^n$ where $n = 3, 4, \dots$ be homeomorphic to S^{n-1} . Assume that A is convex where A is the bounded component of $R^n - X$. Then $\sigma^*(X) \geq \sigma(X)/2$.

Proof. This follows from Theorem 2 and (1).

Corollary 3. Let $X \subset R^n$ where $n = 3, 4, \dots$ be homeomorphic to S^{n-1} . Assume that A is convex where A is the bounded component of $R^n - X$. Then $\sigma_0^*(X) \geq \sigma_0(X)/2$.

Proof. This follows from Theorem 2 and (2).

REFERENCES

1. A. Lelek, *On the surjective span and semispan of connected metric spaces*, Colloq. Math. **37** (1977), 35-45.
2. —, *Continua of constant distances*, Pacific J. Math. **123** (1986), 161-171.
3. J. Munkres, *Elements of algebraic topology*, Addison-Wesley, 1984.
4. T. West, *On the relationships of spans of convex continua*, Proc. Amer. Math. Soc. **111** (1991), 261-265.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTHWESTERN LOUISIANA, LAFAYETTE, LOUISIANA 70504