ON FIRMLY NONEXPANSIVE MAPPINGS

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ABSTRACT. It is shown that any λ -firmly, $0 < \lambda < 1$, nonexpansive mapping $T: C \rightarrow C$ has a fixed point in C whenever C is a finite union of nonempty, bounded, closed convex subsets of a uniformly convex Banach space.

Let C be a nonempty subset of a Banach space X , and let $\lambda \in (0, 1)$. Then a mapping $T: C \rightarrow X$ is said to be λ -firmly nonexpansive if

$$(1) \quad \|Tx - Ty\| \leq \|(1 - \lambda)(x - y) + \lambda(Tx - Ty)\|$$

for all $x, y \in C$. In particular, if (1) holds for every $\lambda \in (0, 1)$, then T is said to be firmly nonexpansive. It is clear that every λ -firmly nonexpansive mapping is nonexpansive; i.e., for each x and y in C , we have $\|Tx - Ty\| \leq \|x - y\|$. Conversely, to each nonexpansive $T: C \rightarrow C$ one can associate a firmly nonexpansive mapping with the same fixed-point set whenever C is closed and convex [3]. Moreover, from the point of view of fixed-point theory for the class of all closed convex subsets C , firmly nonexpansive mappings $T: C \rightarrow C$ do not exhibit better behavior than nonexpansive mappings in general [2]. However, this behavior is completely different in the class of nonconnected subsets C . Indeed, in this note we prove the following result:

Theorem. Let X be a uniformly convex Banach space, let $C = \bigcup_{k=1}^n C_k$ be a union of nonempty, bounded, closed convex subsets C_k of X , and suppose $T: C \rightarrow C$ is λ -firmly nonexpansive for some $\lambda \in (0, 1)$. Then T has a fixed point in C .

This theorem is no longer true if T is merely nonexpansive, even in one-dimensional space $X = \mathbb{R}$. For example, if $C = [-2, -1] \cup [1, 2]$, then the mapping $T: C \rightarrow C$ defined by $Tx = -x$ is nonexpansive and fixed-point free. On the other hand, if $C_1 = \cdots = C_n = C$, then the theorem is true for each nonexpansive mapping $T: C \rightarrow C$, which is the celebrated result of Browder [1], Göhde [4], and Kirk [5]. More generally, the same result is also valid when $\bigcap_{k=1}^n C_k \neq \emptyset$ [6].

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Proof of the theorem. Let $z \in C$, and let x_k be the unique asymptotic center [3] of sequence $(T^i z)$ with respect to the bounded, closed convex subset C_k ($1 \leq k \leq n$). In other words, by the uniform convexity of X , the point $x_k \in C_k$ is uniquely determined by the identity

$$(2) \quad f(x_k) = \inf_{x \in C_k} f(x),$$

where the convex functional $f: X \rightarrow [0, \infty)$ is defined by

$$f(x) = \limsup_{i \rightarrow \infty} \|x - T^i z\|.$$

Since T is nonexpansive, we have $\|Tx_k - T^{i+1}z\| \leq \|x_k - T^i z\|$. Hence

$$(3) \quad f(Tx_k) \leq f(x_k)$$

for all k . Now, if $Tx_k \in C_k$ for some k , then the uniqueness of asymptotic center x_k in conjunction with (2) and (3) yields $Tx_k = x_k$, which completes the proof. Otherwise, if $Tx_k \notin C_k$ for all k , then there exist integers $\{n_1, n_2, \dots, n_m\} \subset \{1, 2, \dots, n\}$ ($m \geq 2$) such that $Tx_{n_k} \in C_{n_{k+1}}$ ($k = 1, \dots, m-1$) and $Tx_{n_m} \in C_{n_1}$. Clearly, without loss of generality, one can rearrange the sequence C_k in such a way that $n_k = k$ for all k . Then we have $Tx_k \in C_{k+1}$ ($k = 1, 2, \dots, m-1$), $Tx_m \in C_1$, and $x_k \in C_k$ for all k . Hence, one can combine (2) and (3) in order to get

$$\begin{aligned} f(x_1) &\leq f(Tx_m) \leq f(x_m) \leq f(Tx_{m-1}) \\ &\leq f(x_{m-1}) \leq \dots \leq f(x_2) \leq f(Tx_1) \leq f(x_1). \end{aligned}$$

Thus we have $f(Tx_k) = f(x_{k+1})$ which, in view of the uniqueness of asymptotic centers x_k , yields

$$(4) \quad x_{k+1} = Tx_k \quad (k = 1, 2, \dots, m),$$

where we have denoted $x_{m+1} = x_1$ for the latter simplicity. Hence we readily derive

$$\|x_{k-1} - x_k\| = \|Tx_k - Tx_{k+1}\| \leq \|x_k - x_{k+1}\| = \|Tx_{k-1} - Tx_k\| \leq \|x_{k-1} - x_k\|$$

for every k , and

$$(5) \quad \|x_1 - x_2\| = \|x_2 - x_3\| = \dots = \|x_{m-1} - x_m\| = \|x_1 - x_m\| := \gamma.$$

Clearly, if $\gamma = 0$, then $x_1 = x_2 = Tx_1$ by (4). Hence the proof is completed in this case. Thus it remains to show that the inequality $\gamma > 0$ is impossible. Since T is λ -firmly nonexpansive for some $\lambda \in (0, 1)$, it follows from (4) that

$$\begin{aligned} \gamma &= \|x_{k+1} - x_k\| = \|Tx_k - Tx_{k-1}\| \leq \|(1-\lambda)(x_k - x_{k-1}) + \lambda(x_{k+1} - x_k)\| \\ &\leq (1-\lambda)\|x_k - x_{k-1}\| + \lambda\|x_{k+1} - x_k\| = \gamma. \end{aligned}$$

Therefore, in view of strict convexity of the norm, we get

$$x_k - x_{k-1} = \alpha_k(x_{k+1} - x_k) \quad (2 \leq k \leq m, x_{m+1} = x_1),$$

for some $\alpha_k \neq 0$. Moreover, by (5) we have $|\alpha_k| = 1$ for all k . Hence the following two cases may occur:

Case I. If $\alpha_k = -1$ for some k , then $x_{k-1} = x_{k+1}$. Hence one can apply (1) and (4) to get

$$\begin{aligned}\|x_k - x_{k-1}\| &= \|Tx_{k-1} - Tx_k\| \leq \|(1 - \lambda)(x_{k-1} - x_k) + \lambda(x_k - x_{k-1})\| \\ &= |1 - 2\lambda| \|x_k - x_{k-1}\| < \|x_k - x_{k-1}\|.\end{aligned}$$

This contradiction shows that $\gamma = 0$.

Case II. If $\alpha_k = 1$ for all k , then we have

$$\begin{aligned}x_2 &= (x_1 + x_3)/2, & x_3 &= (x_2 + x_4)/2, \dots, x_{m-1} = (x_{m-2} + x_m)/2, \\ & & x_m &= (x_1 + x_{m-1})/2.\end{aligned}$$

Clearly, the first $m-2$ identities mean that the points $x_2, x_3, \dots, x_{m-1}$ divide the interval $[x_1, x_m]$ into $m-1$ subintervals $[x_k, x_{k+1}]$ ($1 \leq k \leq m-1$) of the same length $\gamma > 0$. But this leads to a contradiction with the identity $x_m = (x_1 + x_{m-1})/2$. Hence we have always $\gamma = 0$, which completes the proof.

REFERENCES

1. F. E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proc. Nat. Acad. Sci. **54** (1965), 1041-1044.
2. K. Goebel and W. A. Kirk, *Topics in metric fixed point theory*, Cambridge University Press, Cambridge, 1990.
3. K. Goebel and S. Reich, *Uniform convexity, hyperbolic geometry, and nonexpansive mappings*, Marcel Dekker, New York, 1984.
4. D. Göhde, *Zum Prinzip der kontraktiven Abbildung*, Math. Nachr. **30** (1965), 251-258.
5. W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly **72** (1965), 1004-1006.
6. J. Reinermann and R. Schöneberg, *Some results and problems in the fixed point theory for nonexpansive and pseudocontractive mappings in Hilbert space*, Fixed Point Theory Appl. (Srinivasa Swaminathan, ed.) (Proc. Semin. Halifax, 1975), Academic Press, New York, 1976, pp. 187-196.

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