REMARK ON A PAPER OF CHENG AND SMOLLER

VINOD B. GOYAL

(Communicated by Barbara L. Keyfitz)

ABSTRACT. A class of functions $K(r)$ and the corresponding solutions are produced in the case of the differential equation

$$\frac{d^2u}{dr^2} + \frac{1}{r} \frac{du}{dr} + K(r)e^{2u(r)} = 1.$$

The behavior of these solutions at an isolated singularity is also investigated.

In [1] the following question was posed:

Which functions $K(x)$ on S^2 can be Gauss curvature of a metric conformally equivalent to the standard metric on S^2 , or equivalently, which functions $K(x)$ on S^2 are such that the equation

$$\Delta u(x) + K(x)e^{2u(x)} = 1, \quad u \in S^2,$$

admits a solution u ? We consider the case of rotationally symmetric K .

A related problem is to find the functions $K(r)$ which would guarantee the existence of the solutions to

$$(1) \quad \frac{d^2u}{dr^2} + \frac{1}{r} \frac{du}{dr} + K(r)e^{2u} = 1.$$

The purpose of this note is to show that a class of functions $K(r)$ and the corresponding solutions can be produced by means of elementary arguments. The behavior of these solutions at an isolated singularity is also investigated.

First, we change the independent variable from r to t by the substitution $r = e^t$. Since

$$\frac{du}{dr} = e^{-t} \frac{du}{dt}$$

and

$$\frac{d^2u}{dr^2} = e^{-2t} \frac{d^2u}{dt^2} - e^{-2t} \frac{du}{dt},$$

Received by the editors April 30, 1990 and, in revised form, June 1, 1990.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 47G05; Secondary 35S05.

Key words and phrases. Gaussian curvature, conformally equivalent, metric, rotationally symmetric, isolated singularity.

equation (1) becomes

$$e^{-2t} \frac{d^2 u}{dt^2} + K(e^t) e^{2u} = 1$$

or, equivalently,

$$(2) \quad \frac{d^2 u}{dt^2} + e^{2u+2t+\ln K(e^t)} = e^{2t}.$$

Now letting

$$(3) \quad 2u + 2t + \ln K(e^t) = \omega,$$

we get

$$(4) \quad \frac{1}{2} \frac{d^2 \omega}{dt^2} + e^\omega = \frac{1}{2} \frac{d^2}{dt^2} (\ln K(e^t)) + e^{2t}.$$

If we choose K such that

$$\frac{1}{2} \frac{d^2}{dt^2} \ln K(e^t) + e^{2t} = 0,$$

then $K(r) = Dr^c e^{-r^2/2}$ where C and D are arbitrary constants.

With this choice of $K(r)$, equation (4) reduces to

$$(5) \quad \frac{d^2 \omega}{dt^2} + 2e^\omega = 0.$$

To solve this for ω , we multiply by $2 \frac{d\omega}{dt}$ and integrate with respect to t , so that

$$\frac{d\omega}{dt} = 2ie^{\omega/2}.$$

Integrating once again, we get

$$\omega = 2 \ln \left(\frac{1}{E - ti} \right),$$

where E is an arbitrary constant. Now (3) yields

$$(6) \quad u = \ln \left(\frac{e^{r^2/4}}{(E - i \ln r) Gr^{(c+2)/2}} \right),$$

where $G = \sqrt{D}$.

Further it is easy to see that the solutions u are such that

$$\lim_{r \rightarrow 0} \left(\frac{u}{\ln(\frac{1}{r})} \right) = \frac{c+2}{2}.$$

Remarks. As the method shows,

- (1) e^{2u} can be replaced by e^{au} where a is an arbitrary positive constant.
- (2) 1 on the right-hand side of equation (1) can be replaced by any polynomial in r , $\ln r$ or by $r^m \ln r$ ($m > 0$).

(3) If instead of (1), we consider the equation

$$(7) \quad \frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} - K(r)e^{2u} = 1, \quad K(r) > 0,$$

then, as before, $K(r) = Dr^c e^{-r^2/2}$, where C and D are arbitrary positive constants; and the corresponding solutions u are given by

$$u = \ln \left(\frac{e^{r^2/4}}{(E - \ln r)Gr^{(c+2)/2}} \right),$$

where $G = \sqrt{D}$. Clearly, $u \rightarrow \infty$ as $r \rightarrow 0$. Thus, equation (7) does not have entire (i.e. $C^2(E^2)$) solutions, whereas in the case of (1), $u = 0$ if $K(r) = 1$.

REFERENCES

1. *Conformal metrics with prescribed Gaussian curvature on S^2* , UAB International Conference on Differential Equations and Mathematical Physics (Abstracts), March 15–21, 1990, p. 48.

DEPARTMENT OF MATHEMATICS, TUSKEGEE UNIVERSITY, TUSKEGEE, ALABAMA 36088