A CELLULAR WEDGE IN R^3

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ABSTRACT. We construct a cellular wedge $A \vee B \subset R^3$ such that A is not cellular.

In a recent list of problems circulated by R. J. Daverman, the following question [1, E19] is attributed to Morton Brown: If a wedge $A \vee B \subset R^3$ is cellular, is A cellular? (Daverman was able to construct cellular wedges in R^n , $n \geq 5$, having a noncellular constituent. His examples are included in an addendum to this paper.) The following example answers this question negatively.

Example. Let $A \subset R^3$ be the well-known Fox–Artin arc [5, Example 1.3], constructed so as to be locally polyhedral modulo its endpoints p and q . There is a piecewise linear 3-cell whose interior contains p and whose boundary is pierced by A at a single point. Guided by A , we “bore out” a tunnel in the 3-cell pinching down to p , and so construct a crumpled cube B such that B is not a 3-cell, $\text{Bd } B$ is locally polyhedral modulo p , and $A \cap B = \{p\}$; the process of obtaining B is more precisely described in [11]. The arc A is not cellular because, as shown explicitly in [5, Example 1.3], $R^3 - A$ is not an open 3-cell (in particular, Fox and Artin show that there is a simple closed curve in $R^3 - A$ that is not contained in the interior of a 3-cell in $R^3 - A$). On the other hand, $X = A \cup B$ is cellular. The argument for this is well known, but we give a brief sketch here for completeness. To begin, note that there are arbitrarily small 3-cells C such that $B \cap C$ is a disk whose interior contains p and $B \cup C$ is a piecewise linear 3-cell whose boundary is pierced by A at a single point. It is easily seen that $Y = X \cup C$ is cellular by constructing a 3-cell neighborhood of Y in the ε -neighborhood of Y . To accomplish this, begin with a small piecewise linear sphere Σ about q that is disjoint from B and is pierced by A at three points. Let these points be denoted by a_1 , a_2 , and a_3 , ordered moving from p to q . Let D_1 , D_2 , and D_3 be pairwise disjoint small disks on Σ about a_1 , a_2 , and a_3 , respectively. Within the ε -neighborhood of Y , stretch D_1 in

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the direction of p to create a “feeler” that finally “captures” $B \cup C$, creating a sphere (which we continue to call Σ) close to Y whose interior contains all of Y except the subarc of A whose endpoints are a_2 and a_3 . Similarly stretch D_2 , uncovering some of Y this time, to create a sphere Σ close to Y whose interior contains all of Y except the union of $B \cup C$ and the subarc of A whose endpoints are p and a_3 . Finally, stretch D_3 in the direction of p to create a sphere Σ close to Y whose interior contains all of Y . It follows that Y is cellular. Consequently, if U is a neighborhood of X there is a cellular set D , topologically equivalent to Y , such that $X \subset D \subset U$; it follows that there is a 3-cell neighborhood of D (hence of X) in U , and so X is cellular.

Using the arcs of [6], one can obtain uncountably many inequivalently embedded examples constructed such as that of the previous paragraph.

One might hope to begin to extend the example to higher dimensions by first suspending $A \vee B$ and then collapsing the tame arc obtained as the suspension of the wedge point to obtain a wedge in R^4 . It can easily be shown that the suspension of a cell-like set X in R^n , $n \geq 3$, is cellular in R^{n+1} if and only if $R^n - X$ is simply connected. Thus our wedge in R^4 will be cellular; but the portion of it that corresponds to the suspension of A will also be cellular because, as noted in [5], $R^3 - A$ is simply connected. It is natural to try to salvage the process by using, instead of A , the arc of [5, Example 1.1] whose complement in R^3 is not simply connected. That this will *not* give an example is shown by the following.

Theorem 1. *If $A \subset R^3$ is a continuum and $B \subset R^3$ is a crumpled cube such that*

- (1) $B \cap A$ consists of a single point p ,
- (2) $\text{Bd } B - \{p\}$ is locally polyhedral, and
- (3) $A \cup B$ is cellular,

then $R^3 - A$ is simply connected.

Proof. Let J be a loop in $R^3 - A$. Homotop J by a small homotopy to a nonsingular loop J' which is piecewise linear and in general position with respect to $\text{Bd } B - \{p\}$. Since $\text{Bd } B - \{p\}$ is locally polyhedral, $B - \{p\}$ is simply connected [7]. Since in addition $R^3 - (A \cup B)$ is simply connected and $\text{Bd } B - \{p\}$ is path connected, it is easy to construct (via a “Van Kampen” argument) a singular disk in $R^3 - A$ bounded by J' .

It is interesting to note that the techniques used above also establish the following.

Proposition. *If A is an arc in R^3 that is locally tame modulo its endpoints and if there is a tame 2-sphere in R^3 that intersects A transversely in a single point, then $R^3 - A$ is simply connected.*

In our example, B is cellular. The following theorem shows that this is necessary for an example such as this. We say the wedge $A \vee B \subset R^3$ can be

split if there is a closed topological plane $P \subset R^3$ such that $P \cap (A \vee B) = A \cap B$ and P separates $A - B$ from $B - A$. We may assume P is locally polyhedral modulo $A \cap B$. In this case, the closure of at least one of the complementary domains of P is a manifold with boundary. If this closure contains A , we say that $A \vee B$ *splits with A in a manifold half*.

Theorem 2. *Suppose the wedge $A \vee B \subset R^3$ is cellular. Then B is cellular if either*

- (1) *A is one-dimensional,*
- (2) *A can be triangulated so as to collapse to the wedge point, or*
- (3) *$A \vee B$ splits with A in a manifold half.*

Proof. Under condition (1), the result follows from [10, Corollary 4]. Under condition (2), the result follows from the proof of [9, Theorem 1]. If (3) holds, let P be a closed plane, locally polyhedral modulo $\{p\} = A \cap B$, such that P splits $A \vee B = X$ and A is in a manifold half. We will show that B satisfies the cellularity criterion [8]. To this end, let U be a neighborhood of B . We may assume that U is a compact PL manifold with boundary and that $U = M \cup C$ where C is a 3-cell, M is a crumpled 3-manifold, and $M \cap C = P \cap \partial C$ is a disk whose interior contains p . Let N be a compact neighborhood of $\text{Cl}(A - U)$ such that $M \cap N = \emptyset$, and let $U' = U \cup N$. There is a neighborhood V' of X such that each loop in $V' - X$ is nullhomotopic in $U' - X$. Inside V' , construct a compact PL manifold neighborhood V of B such that $V = L \cup D$ where D is a 3-cell, L is a crumpled 3-manifold, $L \subset M$, $D \subset C$, and $L \cap D = P \cap \partial D$ is a disk whose interior contains p . Let α be a loop in $\text{Int } V - B$. We may assume that α is PL and in general position with respect to P . Because we can push the portion of α that lies in D to $D \cap P$, we may assume that α is a loop in $L - B \subset V' - X$. Then α is nullhomotopic in $U' - X$. This nullhomotopy may be “cut off” on the disk $\text{Cl}(\partial C - M \cap C)$ to obtain a nullhomotopy of α in $U - B$.

Addendum. We describe examples, due to R. J. Daverman, of cellular wedges in the n -sphere, $n \geq 5$, each having a noncellular constituent. Let K be a two-dimensional acyclic subcomplex of S^{n-1} , $n \geq 5$, having nontrivial fundamental group. For a discussion of such embedded subcomplexes, see [2, pp. 103–104]. Regard S^n as the suspension of S^{n-1} , and let L denote the subcomplex of S^n determined by coning K from one of the suspension points. We identify L with $(K \times I)/(K \times \{1\})$, and we denote by M the quotient space obtained from S^n by identifying the sets $\{K \times \{t\} \mid 0 \leq t \leq 1/2\}$ with points. The Sum Theorem for dimension implies that M is finite-dimensional, and it follows from [4, Corollary 5.2] that M is a generalized n -manifold. By [2, Corollary 2A, p. 287] M has a resolution, and so by [2, Theorem 3, p. 181] M will be a manifold (hence topologically S^n) provided M can be shown to have the disjoint disks property. This latter property may be verified much as in the proof of [3, Main Lemma]. In M , the image of L is topologically the wedge

of an arc and the suspension of K ; the wedge is cellular since its preimage in S^n is cellular, while the arc is not since its preimage in S^n (being not simply connected) is not.

REFERENCES

1. Robert J. Daverman, *Problems about finite-dimensional manifolds*, preprint.
2. —, *Decompositions of manifolds*, Pure Appl. Math., vol. 124, Academic Press, Orlando, 1986.
3. —, *Embedding phenomena based upon decomposition theory: wild Cantor sets satisfying strong homogeneity properties*, Proc. Amer. Math. Soc. **75** (1979), 177–182.
4. R. J. Daverman and J. J. Walsh, *Acyclic decompositions of manifolds*, Pacific J. Math. **109** (1983), 291–303.
5. Ralph H. Fox and Emil Artin, *Some wild cells and spheres in three-dimensional space*, Ann. of Math. (2) **49** (1948), 979–990.
6. C. H. Giffen, *Uncountably many inequivalent nearly tame arcs*, Notices Amer. Math. Soc. **10** (1963), 666–667.
7. O. G. Harrold, Jr. and E. E. Moise, *Almost locally polyhedral spheres*, Ann. of Math. (2) **57** (1953), 575–578.
8. D. R. McMillan, Jr., *A criterion for cellularity in a manifold*, Ann. of Math. (2) **79** (1964), 327–337.
9. —, *Piercing a disk along a cellular set*, Proc. Amer. Math. Soc. **19** (1968), 153–157.
10. R. B. Sher, *Defining subsets of E^3 by cubes*, Pacific J. Math. **25** (1968), 613–619.
11. C. D. Sikkema, *A duality between certain spheres and arcs in S^3* , Trans. Amer. Math. Soc. **122** (1966), 399–415.

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