MULTI-TUPLE SHILOV BOUNDARIES OF ALGEBRA TENSOR PRODUCTS

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ABSTRACT. We give a short proof of Basener Slodkowski theorem for multi-tuple Shilov boundaries of tensor products of two uniform algebras.

1. INTRODUCTION

Since their discovery in 1975 by Basener and Sibony the multi-tuple Shilov boundaries of commutative Banach algebras proved to be an essential tool in the investigating multi-dimensional analytic structures in algebra spectra. Results concerning the relationship of these boundaries and analyticity in algebra spectra have appeared often during the last fifteen years (e.g. Basener [1], Kramm [3], Kumagai [4], Sibony [5], Tonev [10, 11], etc.). Various properties of multi-dimensional Shilov boundaries were investigated by Basener [1], Sibony [5], Slodkowski [6, 7], Tonev [9, 10] and others.

In 1983 Slodkowski [6] proved a conjecture originally due to Basener about expression of multi-tuple Shilov boundaries of a tensor product of two algebras in terms of multi-tuple Shilov boundaries of generating algebras. Namely:

Basener Slodkowski Theorem [7]. $\partial^{(n)} A \otimes B = \bigcup_{k=1}^n \partial^{(k)} A \times \partial^{(n-k-1)} B$.

This theorem generalizes the well known fact about the Shilov boundary of a tensor product of two algebras and also the corresponding result about multi-tuple Shilov boundaries of polydisc algebras. Applications of this conjecture were pointed out by Kumagai [4]. Basener has proved the theorem for certain special cases only. Slodkowski's proof utilizes the theory of q -plurisubharmonic functions and a specific approximation procedure of algebra elements. It turns out that the notion of multi-tuple Shilov boundaries can be extended for function spaces as well. The first part of Slodkowski's proof in fact proves Basener's conjecture for the space $P(X)$, the closure of polynomials in $\mathbb{C}^n$ on a compact set $X \in \mathbb{C}^n$.

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2. PRELIMINARIES

Let A be a *uniform algebra* over $\mathbb{C}$ with unit. That is A is a separating closed subalgebra of the space of all continuous complex valued functions on some compact Hausdorff space X which contains the constants and the norm of $f \in A$ is the maximum of $|f(x)|$ on X . As usual $\text{sp } A$ denotes the *maximal ideal space* of A and $\hat{f}$ denotes the *Gelfand extension* of a given function of A . The *Shilov boundary* ∂A of A is the smallest closed subset of $\text{sp } A$ on which the Gelfand extensions of all functions of A assume the maximums of their absolute values.

Denote by $V(f_1, \dots, f_n)$ the vanishing set of a fixed n -tuple $(\hat{f}_1, \dots, \hat{f}_n)$ over $\hat{A}$, i.e.

$$V(f_1, \dots, f_n) = \{m \in \text{sp } A : \hat{f}_1(m) = \hat{f}_2(m) = \dots = \hat{f}_n(m) = 0\}.$$

By A_E we denote the closure in $C(E)$ of restrictions of all elements of $\hat{A}$ on a fixed closed subset E of $\text{sp } A$.

Definition 1 (Basener, Sibony). The n -tuple Shilov boundary $\partial^{(n)} A$ of a commutative Banach algebra A is the following subset of $\text{sp } A$:

$$\partial^{(n)} A = \left[\bigcup \{ \partial A_{V(f_1, \dots, f_{n-1})} : (f_1, \dots, f_{n-1}) \in A^{n-1} \} \right]$$

where $[N]$ stands for the closure of N in $\text{sp } A$, $A^0 = \{0\}$, and A^n is the set of all n -tuples of elements of A for $n \geq 1$ (see [1, 5], also [12]).

It is easy to check that

$$\partial A = \partial^{(1)} A \subset \partial^{(2)} A \subset \partial^{(3)} A \subset \dots \subset \partial^{(n)} A \subset \dots \subset \text{sp } A.$$

We recall some of the basic properties of multi-tuple Shilov boundaries. An n -tuple $F = (\hat{f}_1, \dots, \hat{f}_n) \in \hat{A}^n$ is called *regular* if the functions $\hat{f}_1, \dots, \hat{f}_n$ have no common zeros on $\text{sp } A$, i.e. if $V(f_1, \dots, f_n) = \emptyset$ or, in other words, if $0 \notin (\hat{f}_1, \dots, \hat{f}_n)(\text{sp } A)$.

Theorem 1 [9, Theorem 1]. $\partial^{(n)} A$ is the smallest closed subset of $\text{sp } A$ on which every regular n -tuple $F = (\hat{f}_1, \dots, \hat{f}_n) \in \hat{A}^n$ assumes the minimum of its norm

$$(1) \quad \|F(m)\| = \left(\sum_{j=1}^n \|\hat{f}_j(m)\|^2 \right)^{1/2}, \quad m \in \text{sp } A.$$

Given a subset K of $\mathbb{C}^n$, bK will denote in the sequel the topological boundary of K with respect to the ambient $\mathbb{C}^n$.

Theorem 2 [9, Theorem 3]. $\partial^{(n)} A$ is the smallest closed subset of $\text{sp } A$ such that the inclusion

$$(2) \quad F(\partial^{(n)} A) \supset bF(\text{sp } A)$$

holds for every n -tuple $F = (\hat{f}_1, \dots, \hat{f}_n) \in \hat{A}^n$.

3. BASENER-SŁODKOWSKI THEOREM

Let A and B be two uniform algebras and let $A \otimes B$ denote the tensor product of A and B . In other words $A \otimes B$ is isometrically isomorphic to the smallest closed subalgebra of $C(\operatorname{sp} A \times \operatorname{sp} B)$ which contains all monomials $f(x)g(y)$, where $f \in \widehat{A}$, $g \in \widehat{B}$, $x \in \operatorname{sp} A$, $y \in \operatorname{sp} B$. It is not hard to see that $\operatorname{sp} A \otimes B$ is homeomorphic to the set $\operatorname{sp} A \times \operatorname{sp} B$ and that

$$(3) \quad \partial A \otimes B = \partial A \times \partial B.$$

The theorem under consideration may be stated as follows.

Theorem 3 (Basener-Słodkowski). $\partial^{(n)} A \otimes B = \bigcup_{k=1}^n \partial^{(k)} A \times \partial^{(n-k+1)} B$.

Proof. For the sake of simplicity we shall assume that $A = \widehat{A}$ and $B = \widehat{B}$. If we denote $\operatorname{sp} A$ by X and $\operatorname{sp} B$ by Y , then $\operatorname{sp} A \otimes B$ is homeomorphic to $X \times Y$. The inclusion

$$(4) \quad \partial^{(n)} A \otimes B \supset \bigcup \partial^{(k)} A \times \partial^{(n-k+1)} B$$

is the easier one and can be checked without any trouble. The opposite inclusion

$$(5) \quad \partial^{(n)} A \otimes B \subset \bigcup \partial^{(k)} A \times \partial^{(n-k+1)} B$$

we shall prove in two steps. Denote for a while the set $\bigcup \partial^{(k)} A \times \partial^{(n-k+1)} B$ by E .

Step 1. Any function of type

$$(6) \quad F(x, y) = (f_1(x), \dots, f_k(x), g_1(y), \dots, g_{n-k}(y)),$$

where $f_i(x) \in A$ and $g_j(y) \in B$, $i = 1, \dots, k$, $j = 1, \dots, n - k$, satisfies the following inclusion

$$(7) \quad bF(X \times Y) \subset F(E).$$

Indeed, let $(x_0, y_0) \in X \times Y$ and let $\|F(x, y)\| \neq 0$. Since

$$\begin{aligned} \|F(x, y)\|^2 &= \|(f_1(x), \dots, f_k(x), g_1(y), \dots, g_{n-k}(y))\|^2 \\ &= \|(f_1(x), \dots, f_k(x))\|^2 + \|(g_1(y), \dots, g_{n-k}(y))\|^2 \end{aligned}$$

we have that

$$\begin{aligned} \|F(x_0, y_0)\|^2 &\geq \min_{X \times Y} \|F(x, y)\|^2 \\ &= \min_X \|(f_1(x), \dots, f_k(x))\|^2 + \min_Y \|(g_1(y), \dots, g_{n-k}(y))\|^2. \end{aligned}$$

If $\|(g_1(y), \dots, g_{n-k}(y))\|$ vanishes on Y then certainly the k -tuple $(f_1(x), \dots, f_k(x))$ does not vanish on X and

$$\min_X \|(f_1(x), \dots, f_k(x))\| = \min_{\partial^{(k)} A} \|(f_1(x), \dots, f_k(x))\|$$

by Theorem 1, so that

$$\begin{aligned}\|F(x_0, y_0)\| &\geq \min_{\partial^{(k)} A \times V(g_1, \dots, g_{n-k})} \|F(x, y)\| \\ &= \min_{\partial^{(k)} A \times \partial B_{V(g_1, \dots, g_{n-k})}} \|F(x, y)\| \geq \min_{\partial^{(k)} A \times \partial^{(n-k+1)} B} \|F(x, y)\|.\end{aligned}$$

If $\|(f_1(x), \dots, f_k(x))\|$ vanishes on X , then certainly $(g_1(y), \dots, g_{n-k}(y))$ does not vanish on Y and we can obtain in a similar way that $\|F(x_0, y_0)\| \geq \min_{\partial^{(k+1)} A \times \partial^{(n-k)} B} \|F(x, y)\|$. If both $(f_1(x), \dots, f_k(x))$ and $(g_1(y), \dots, g_{n-k}(y))$ do not vanish, then again as above we get that

$$\|F(x_0, y_0)\| \geq \min_{\partial^{(k)} A \times \partial^{(n-k)} B} \|F(x, y)\| \geq \min_{\partial^{(k)} A \times \partial^{(n-k+1)} B} \|F(x, y)\|.$$

Consequently

$$(8) \quad \|F(x_0, y_0)\| \geq \min_{\bigcup_{k=1}^n \partial^{(k)} A \times \partial^{(n-k+1)} B} \|F(x, y)\| = \min_E \|F(x, y)\|$$

for any regular n -tuple F of type (6). If we assume that (7) is violated for some F of type (6) we can find a point $(x_0, y_0) \in X \times Y$ such that $F(x_0, y_0) \in bF(X \times Y) \setminus F(E)$. Consequently we can find a point Z_0 in $C^n \setminus \sigma(F)$ close enough to $F(x_0, y_0)$, such that the function $H = F - Z_0$ does not vanish on $X \times Y$ and at the same time does not satisfy the inequality (8) in contradiction with the result just proved. We conclude that the inclusion (7) is fulfilled for any n -tuple F of type (6).

Step 2. The inclusion (7) holds for every mapping $F(x, y) \in (A \otimes B)^2$.

Proof. We prove the case $n = 2$ of the statement. Let $F \in A^k$, $g \in B^l$ and let P_1 and P_2 be polynomials in k and l variables respectively. It is not difficult to check that

$$\partial^{(j)} P(F(X)) \subset F(\partial^{(j)} P(X))$$

for each j and for every $F \in A^k$. Therefore

$$\begin{aligned}b(P_1(F, G), P_2(F, G))(X \times Y) &= b(P_1, P_2)(F(X) \times G(Y)) \\ &\subset (P_1, P_2)(\partial^{(2)} P(F(X) \times G(Y))) \\ &= (P_1, P_2)\{(\partial P(F(X)) \times \partial^{(2)} P(G(Y))) \cup (\partial^{(2)} P(F(X)) \times \partial P(G(Y)))\} \\ &\subset (P_1, P_2)\{(F(\partial A) \times G(\partial^{(2)} B)) \cup (F(\partial^{(2)} A) \times G(\partial B))\} \\ &= (P_1, P_2)\{(F, G)(\partial A \times \partial^{(2)} B) \cup (F, G)(\partial^{(2)} A \times \partial B)\} \\ &= (P_1, P_2)(F, G)(\partial A \times \partial^{(2)} B \cup \partial^{(2)} A \times \partial B) \\ &= (P_1, P_2)(F, G)(E) = (P_1(F, G), P_2(F, G))(E).\end{aligned}$$

Since the mappings of type $(P_1(F, G), P_2(F, G))$ are uniformly dense in $(A \otimes B)^2$ we conclude that (7) holds for every function in $(A \otimes B)^2$. This proves the case $n = 2$ of Theorem 3 because by Theorem 2, $\partial^{(2)} A \otimes B$ is the

smallest closed subset of $X \times Y$ for which (7) holds for any mapping from $(A \otimes B)^2$.

The proof of the general case ($n > 2$) of Theorem 3 follows the same lines as the proof of the case $n = 2$.

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