PURE STATE EXTENSIONS AND COMPRESSIBILITY OF THE l_1 -ALGEBRA

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. In 1958 Kadison and Singer proved that not every pure state on a continuous maximal abelian subalgebra (masa) of the algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on a separable complex Hilbert space $\mathcal{H}$, has a unique pure state extension to $\mathcal{B}(\mathcal{H})$ [5]. They conjectured the same result is true for discrete masas, and although the question remains open, it was shown by Anderson in 1978 to be equivalent to determining whether all operators in $\mathcal{B}(\mathcal{H})$ are compressible.

We define in this paper the l_1 -subalgebra $\mathcal{M}$ of $\mathcal{B}(\mathcal{H})$, and show that all operators in $\mathcal{M}$ are compressible. Hence every pure state on a discrete masa has a unique pure state extension to $\mathcal{M}$.

INTRODUCTION

Let $\mathbf{N}$ denote the set of positive integers. Fix an orthonormal basis $\{e_n : n \in \mathbf{N}\}$ for the Hilbert space $\mathcal{H}$. A discrete masa in $\mathcal{B}(\mathcal{H})$ is unitarily equivalent to $l^\infty(\mathbf{N})$, hence to the C^* -subalgebra $\mathcal{D}$ of the diagonal operators in $\mathcal{B}(\mathcal{H})$ with respect to the fixed basis.

The extension problem is: Does every pure state on $\mathcal{D}$ have a unique pure state extension to $\mathcal{B}(\mathcal{H})$? Note that a pure state has a unique state extension if and only if it has a unique pure state extension [2, 2.10.1]. Since $\mathcal{D}$ is abelian, the pure states on $\mathcal{D}$ are the nonzero multiplicative linear functionals on $\mathcal{D}$ [2, 2.5.2].

There is a 1-1 correspondence between pure states on $\mathcal{D}$ and ultrafilters on $\mathbf{N}$, the vector states corresponding to fixed ultrafilters. To see the correspondence, first note that diagonal projections are of the form P_σ for some $\sigma \subseteq \mathbf{N}$, where P_σ denotes the orthogonal projection in $\mathcal{B}(\mathcal{H})$ onto the closed linear span of $\{e_n : n \in \sigma\}$, and that their linear span is dense in $\mathcal{D}$. Any pure state on $\mathcal{D}$, being multiplicative, will map such a projection to 0 or 1. Now, the

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Received by the editors April 17, 1990 and, in revised form, June 1, 1990.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 46L05, 46L30; Secondary 47A20.

Key words and phrases. Pure state, pure state extension, maximal abelian subalgebra, compression, l_1 -algebra, ultrafilter, $\beta(\mathbf{N})$.

relation $f(P_\sigma) = 1$ iff $\sigma \in \mathcal{U}$ establishes the correspondence.

1. COMPRESSIBILITY

In [1] Anderson shows that every pure state on $\mathcal{D}$ has a unique pure state extension to $\mathcal{B}(\mathcal{H})$ if and only if all operators are compressible. Before we give the definition of compressibility, let us associate with every operator T , the diagonal operator D_T given by $\langle D_T e_i, e_i \rangle = \langle T e_i, e_i \rangle$, where $i \in \mathbb{N}$. An operator T is *compressible* if given any ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and any $\varepsilon > 0$, there is a set $\sigma \in \mathcal{U}$ such that $\|P_\sigma(T - D_T)P_\sigma\| < \varepsilon$.

In [1] Anderson proves that this statement is equivalent to the following one which does not involve ultrafilters:

Proposition 1.1. *An operator T is compressible if and only if for every $\varepsilon > 0$, there is a finite partition of $\mathbb{N}$ into $\sigma_1, \dots, \sigma_n$ such that for all $i \leq n$, $\|P_{\sigma_i}(T - D_T)P_{\sigma_i}\| < \varepsilon$.*

We submit a new proof of this result:

Proof. Suppose first that for $\varepsilon > 0$ there is a partition of $\mathbb{N}$ into $\sigma_1, \dots, \sigma_n$ such that for all $i \leq n$, $\|P_{\sigma_i}(T - D_T)P_{\sigma_i}\| < \varepsilon$. Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$.

Whenever a finite union of sets belongs to an ultrafilter, one of the sets must be in the ultrafilter. Thus, there is $j \leq n$ such that $\sigma_j \in \mathcal{U}$. Hence, T is compressible.

Conversely, we will show that if there exists some $\varepsilon > 0$ for which no such partition exists, then there is an ultrafilter that does not compress T . Without loss of generality, assume $D_T = 0$.

Consider the set $\mathcal{F} = \{S \subset \mathbb{N} : \text{there is a partition of } S \text{ into } \sigma_1, \dots, \sigma_n, \text{ such that for all } i \leq n, \|P_{\sigma_i} T P_{\sigma_i}\| < \varepsilon\}$.

Claim. $\mathcal{F}$ is an ideal in the Boolean algebra of subsets of $\mathbb{N}$.

The proof of the claim is as follows:

- (i) $\mathbb{N} \notin \mathcal{F}$, by hypothesis.
- (ii) Let $S \in \mathcal{F}$ and $S' \subseteq S$; consider $\{\sigma_1 \cap S', \dots, \sigma_n \cap S'\}$; then for all $i \leq n$, $\|P_{\sigma_i \cap S'} T P_{\sigma_i \cap S'}\| \leq \|P_{\sigma_i} T P_{\sigma_i}\| < \varepsilon$. So $S' \in \mathcal{F}$.
- (iii) Let $S_1 \in \mathcal{F}$, $S_2 \in \mathcal{F}$. We want to show that the union of S_1 and S_2 is in $\mathcal{F}$. We may assume $S_1 \cap S_2 = \emptyset$ because $S_1 \cup S_2 = S_1 \cup (S_2 - S_1)$, and by (ii) we have $S_2 \in \mathcal{F} \Rightarrow S_2 - S_1 \in \mathcal{F}$. Take as the partition of $S_1 \cup S_2$, the union of the two partitions corresponding to S_1 , and S_2 . This partition has the desired properties.

Now, let $\mathcal{F}^*$ be the dual filter of $\mathcal{F}$, that is, the set of the complements of sets in $\mathcal{F}$, and $\mathcal{U}$ an ultrafilter extending $\mathcal{F}^*$. Any subset σ of $\mathbb{N}$ with $\|P_\sigma T P_\sigma\| \leq \varepsilon$ is in $\mathcal{F}$, so its complement is in $\mathcal{F}^*$, hence in $\mathcal{U}$; therefore $\sigma \notin \mathcal{U}$.

So, for all $\sigma \in \mathcal{U}$, $\|P_\sigma T P_\sigma\| \geq \varepsilon$. Hence $\mathcal{U}$ does not compress T . $\square$

The following proposition reduces the extension problem to an even more elementary problem (see also [4]), a familiar version to the experts in recent

years:

Definition. Let $n \in \mathbb{N}$, and $\delta \geq 0$; an operator T in $\mathcal{B}(\mathcal{H})$ is n -compressible to δ if there is a partition $\sigma_1, \dots, \sigma_n$ of $\mathbb{N}$ such that for all $i \leq n$, $\|P_{\sigma_i}(T - D_T)P_{\sigma_i}\| \leq \delta$.

Proposition 1.2. $\mathcal{B}(\mathcal{H})$ is compressible if and only if there is an integer n , such that every operator of norm 1, on a finite-dimensional Hilbert space, is n -compressible to $\frac{1}{2}$.

Proof. We will show that the following four statements are equivalent:

- (1) $\mathcal{B}(\mathcal{H})$ is compressible.
- (2) Every operator of norm 1 is n -compressible to $\frac{1}{2}$, for some $n \in \mathbb{N}$.
- (3) There is an integer n , such that every operator of norm 1 is n -compressible to $\frac{1}{2}$.
- (4) There is an integer n , such that every operator of norm 1, on a finite-dimensional Hilbert space, is n -compressible to $\frac{1}{2}$.

• (1) $\Leftrightarrow$ (2). (1) $\Rightarrow$ (2): clear, let $\varepsilon = \frac{1}{2}$.

(2) $\Rightarrow$ (1): by iteration and Proposition (1.1).

• (2) $\Leftrightarrow$ (3). (2) $\Rightarrow$ (3): suppose that $\forall n \exists T_n$ such that T_n is not n -compressible to $\frac{1}{2}$; then $T = T_1 \oplus \dots \oplus T_n \oplus \dots$ would not be compressible for any n , which contradicts (2).

(3) $\Rightarrow$ (2): clear.

• (3) $\Leftrightarrow$ (4). (3) $\Rightarrow$ (4): clear.

(4) $\Rightarrow$ (3). Let T be an operator of norm 1. Replacing T by $T - D_T$ if necessary, we may assume that $D_T = 0$. Denote by T_r the restriction of T to the linear span of $\{e_1, \dots, e_r\}$. Let us say that a partition of $\{1, \dots, r\}$ into k pieces $\sigma_1, \dots, \sigma_k$ where $k \leq n$ is *good for r* if $\|P_{\sigma_i}T_rP_{\sigma_i}\| \leq \frac{1}{2}$, for $i = 1, \dots, k$. Our hypothesis implies that there is at least one good partition for each r . Next define a tree as follows. An element of the tree at level r will be a good partition of $\{1, \dots, r\}$. A partition Π_r at level r is joined to a partition $\Pi_{r+1} = \{\sigma_1, \dots, \sigma_k\}$ at level $r+1$ if and only if $\sigma_i \upharpoonright r$ is in Π_r or else $\sigma_i \upharpoonright r$ is empty, where $\sigma_i \upharpoonright r$ denotes the set $\sigma_i \cap \{1, \dots, r\}$, for $i = 1, \dots, k$. In this case we write $\Pi_r \leq \Pi_{r+1}$.

It is clear that each partition at level $r+1$ has a unique predecessor; namely, its restriction to $\{1, \dots, r\}$. Hence we have a tree, and it has infinitely many elements. Also, it is finitely branching, that is, every element has finitely many immediate successors. In fact, if Π_{r+1} is an element of the tree at level $r+1$, and if Π_r is a partition of $\{1, \dots, r\}$ such that $\Pi_r \leq \Pi_{r+1}$, then Π_r must be an element of the tree at level r , since $\|P_{\sigma}T_rP_{\sigma}\| \leq \|P_{\sigma'}T_{r+1}P_{\sigma'}\|$ whenever σ is a subset of σ' . By König's infinity lemma [6, Chapter 2, Lemma 5.7; or 3, Chapter 7, Theorem 1.2], any finitely branching infinite tree has an infinite branch. Let $\Pi_1 \leq \Pi_2 \leq \dots$ be such a branch. Now define a partition $\sigma_1, \dots, \sigma_n$ of $\mathbb{N}$ by letting two integers be equivalent iff they are equivalent for some Π_j , $j \in \mathbb{N}$ (and hence thereafter). The new partition has the property

that for all r , there is an integer $s > r$ so that

$$\{\sigma_1 \upharpoonright r, \dots, \sigma_n \upharpoonright r\} = \{\sigma_1^s \upharpoonright r, \dots, \sigma_n^s \upharpoonright r\},$$

where $\{\sigma_1^s, \dots, \sigma_n^s\}$ is the partition Π_s of our branch. Let us show that this partition satisfies the desired compression property for T . Note that $\|P_{\sigma_i} T P_{\sigma_i}\| = \lim_{r \rightarrow \infty} \|P_r P_{\sigma_i} T P_{\sigma_i} P_r\|$. For any $r \in \mathbb{N}$, we have by the previous remark, an $s > r$ such that $P_r P_{\sigma_i} = P_{\sigma_i \upharpoonright r} = P_{\sigma_i^s \upharpoonright r}$. But $P_{\sigma_i^s \upharpoonright r} T P_{\sigma_i^s \upharpoonright r} = P_{\sigma_i \upharpoonright r} T_r P_{\sigma_i \upharpoonright r}$. Hence for all $i \leq n$, $\|P_{\sigma_i} T P_{\sigma_i}\| \leq \frac{1}{2}$, as wanted. $\square$

Remarks. (1) Note that the proof that (4) implies (3) in Proposition 1.2 actually shows that an operator T is n -compressible to $\frac{1}{2}$ if and only if each of its restrictions T_r is n -compressible to $\frac{1}{2}$.

(2) Although König's infinity lemma is quite intuitive, using the Axiom of Choice, it does not generalize to uncountable trees; in fact there exists an uncountable tree, all of whose levels are countable, which has no uncountable branches (such a tree is called an Aronsajn tree), [3, Chapter 7, Theorem 3.3].

2. THE l_1 -ALGEBRA

Given our orthonormal basis in $\mathcal{H}$, we will define in this section a certain class of matrices, which are bounded and form a $*$ -algebra in $\mathcal{B}(\mathcal{H})$. We will conclude by proving that the norm-closure $\mathcal{M}$ is a proper C^* -algebra of $\mathcal{B}(\mathcal{H})$ which is compressible. Hence every pure state on the diagonal operators extends uniquely to a pure state on $\mathcal{M}$.

Definition. Let M be a matrix with complex entries M_{ij} , $i, j \in \mathbb{N}$. We will say that M is l_1 -bounded if there is a real number m such that

for all $i \in \mathbb{N}$, $\sum_{j \in \mathbb{N}} |M_{ij}| \leq m$, and

for all $j \in \mathbb{N}$, $\sum_{i \in \mathbb{N}} |M_{ij}| \leq m$.

The collection $\mathcal{M}_0$ of all l_1 -bounded matrices is a selfadjoint algebra in $\mathcal{B}(\mathcal{H})$. Moreover, the norm of the operator defined by an l_1 -bounded matrix is no greater than its l_1 -bound [7, Chapter 4]. $\mathcal{M}_0$ is not norm-closed (for example, if T_n is defined for each $n \in \mathbb{N}$ by $T_n(e_1) = \sum_{i=1}^n e_i/i$ and $T_n(e_j) = 0$ for $j \neq 1$, then $\{T_n\}$ is a sequence in $\mathcal{M}_0$ whose norm-limit is not in $\mathcal{M}_0$). Let $\mathcal{M}$ denote the norm-closure of $\mathcal{M}_0$. $\mathcal{M}$ clearly contains all finite rank operators, hence all compact operators. Also it contains the subpermutation matrices (i.e., matrices with at most one nonzero entry in each row and column, that entry being 1). In particular it contains the unilateral shift and the permutation matrices. It also contains the diagonal operators, and hence the C^* -algebra generated by the diagonal operators and the subpermutations [7, Chapter 3]. However $\mathcal{M}$ does not contain all compressible operators, in fact

the isometry defined by

$$V = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1/\sqrt{2} & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1/\sqrt{2} & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1/\sqrt{3} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1/\sqrt{3} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1/\sqrt{3} & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is not in $\mathcal{M}$. To see this, let M be in $\mathcal{M}_0$, with l_1 -bound m ; we have

$$\begin{aligned} \|V - M\|^2 &\geq \|(V - M)e_n\|^2 \geq \sum_{i=1}^n \left| \frac{1}{\sqrt{n}} - t_i \right|^2 \\ &\geq 1 + (|t_1|^2 + \cdots + |t_n|^2) - \frac{2}{\sqrt{n}}(\operatorname{Re}(t_1) + \cdots + \operatorname{Re}(t_n)) \\ &\geq 1 - \frac{2}{\sqrt{n}}m, \end{aligned}$$

where $t_1, \dots, t_n$ are the entries of M corresponding to the nonzero entries of Ve_n . Hence $\|V - M\| \geq 1$. Note that V is also a compressible operator [7].

Our next goal is to show that $\mathcal{M}$ is compressible. Before presenting the proof, it is convenient to introduce some notation and establish some facts.

Let M be a symmetric $(n \times n)$ -matrix with 0 diagonal and nonnegative entries elsewhere.

A *splitting* of M is a partition of $\{1, \dots, n\}$ into two sets σ_1, σ_2 satisfying for each $i \leq n$,

$$\sum_{j \in \sigma_1} M_{ij} \leq \sum_{j \in \sigma_2} M_{ij}, \text{ if } i \in \sigma_1 \quad \text{and} \quad \sum_{j \in \sigma_2} M_{ij} \leq \sum_{j \in \sigma_1} M_{ij}, \text{ if } i \in \sigma_2.$$

Given a partition $\{\sigma_1, \sigma_2\}$ of $\{1, \dots, n\}$, define the *inner sum* of i , $i \leq n$, to be the sum $\sum_{j \in \sigma_1} M_{ij}$ if $i \in \sigma_1$ and $\sum_{j \in \sigma_2} M_{ij}$ if $i \in \sigma_2$. Similarly, define the *outer sum* of i to be the sum $\sum_{j \in \sigma_2} M_{ij}$ if $i \in \sigma_1$ and $\sum_{j \in \sigma_1} M_{ij}$ if $i \in \sigma_2$.

Denote by $p(\sigma_1, \sigma_2)(i)$ the difference between the inner sum of i and the outer sum of i , relative to $\{\sigma_1, \sigma_2\}$. When no confusion arises, we will write p_i .

Define the *potential* of $\{\sigma_1, \sigma_2\}$, $\Pi(\sigma_1, \sigma_2)$ to be $\sum_{i \leq n} p_i$. Basically, a partition will endow each entry of the matrix with a sign, and the potential is given by the sum of those signed entries.

We want to show that every matrix as above has a splitting, which amounts to showing that there is a partition of $\{1, \dots, n\}$ such that for all $i \leq n$, $p_i \leq 0$. Consider a partition $\sigma_1 = \{1, \dots, k-1\}$ and $\sigma_2 = \{k, \dots, n\}$, where $1 \leq k \leq n$. We compute in the next lemma the change in potential when k is shifted from σ_2 to σ_1 .

Lemma 2.1. *We have*

$$\Pi(\sigma_1, \sigma_2) - \Pi(\sigma_1 \cup \{k\}, \sigma_2 - \{k\}) = 4p_k(\sigma_1, \sigma_2).$$

Proof. Let us notice that all the signed entries of M with respect to the partition $\{\sigma_1, \sigma_2\}$ remain the same in the improved partition except for the entries in the k th row and column, where the signs are reversed, as seen in the picture below:

$$\begin{matrix} & & & k & & & \\ \left(\begin{array}{ccccccc} + & \cdots & + & -/+ & - & \cdots & - \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ + & \cdots & + & -/+ & - & \cdots & - \\ -/+ & \cdots & -/+ & + & +/- & \cdots & +/- \\ - & \cdots & - & +/- & + & \cdots & + \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ - & \cdots & - & +/- & + & \cdots & + \end{array} \right) \end{matrix}$$

By an easy computation we get

$$\begin{aligned} \Pi(\sigma_1, \sigma_2) - \Pi(\sigma_1 \cup \{k\}, \sigma_2 - \{k\}) &= 4 \left(\sum_{i=k+1}^n M_{ik} - \sum_{i=1}^{k-1} M_{ik} \right) \\ &= 4p_k(\sigma_1, \sigma_2). \quad \square \end{aligned}$$

Proposition 2.2. *Every finite symmetric nonnegative matrix with 0 on the diagonal has a splitting.*

Proof. Let M be such an $(n \times n)$ -matrix. Let us start with a partition of $\{1, \dots, n\}$ into two sets $\sigma_1 = \{1, \dots, k-1\}$ and $\sigma_2 = \{k, \dots, n\}$, where $1 \leq k \leq n$ and $p_k > 0$. The idea is to “improve” the partition by shifting k from σ_2 into σ_1 . By Lemma 2.1, a shift strictly decreases the potential. Since there are only finitely many possible partitions of $\{1, \dots, n\}$ into two sets, there is a partition $\{\sigma_1, \sigma_2\}$ with minimal potential. If $p_k(\sigma_1, \sigma_2) > 0$ for some k , then by Lemma 2.1, this partition would not have minimal potential. Hence $p_k(\sigma_1, \sigma_2) \leq 0$ for all k , and so $\{\sigma_1, \sigma_2\}$ is a splitting. $\square$

Theorem 2.3. $\mathcal{M}$ is a compressible C^* -algebra in $\mathcal{B}(\mathcal{H})$.

Proof. Let us first remark that, since the collection of compressible operators is norm-closed [7], it is enough to show that $\mathcal{M}_0$ is compressible. Fix a selfadjoint element M in $\mathcal{M}_0$ with l_1 -bound m and 0 diagonal. Let A denote the matrix whose entries are the absolute values of the entries for M . We have that A is symmetric, nonnegative, and also has l_1 -bound m . Denote by A_r (resp. M_r) the restriction of A (resp. M) to the linear span of $\{e_1, \dots, e_r\}$. By Proposition 2.2, there is a splitting $\{\sigma_1, \sigma_2\}$ for A_r . Since each inner sum is no greater than each outer sum, the l_1 -bounds of $P_{\sigma_i} A_r P_{\sigma_i}$, $i = 1, 2$, are no greater than $m/2$. By applying the proposition sufficiently many times, we can get a partition of $\{1, \dots, r\}$ where $P_{\sigma} A_r P_{\sigma}$ has l_1 -bound no greater than $\frac{1}{2}$,

for each σ in the partition. But M_r and A_r have the same l_1 -bound, hence $\|P_\sigma M P_\sigma\| \leq \frac{1}{2}$. Thus M_r is n -compressible to $\frac{1}{2}$ for some n , and by the remark following Proposition 1.2, so is M .

Now, let M be any element in $\mathcal{M}_0$. By replacing M by $M - D_M$ if necessary, we may assume that the diagonal of M is 0. By the argument above, the real and imaginary parts of M are each compressible. Since l_1 -compressible operators form a linear space, M is also compressible. $\square$

Finally, we would like to mention some more general results concerning the extension problem that have been proved in the last few years. Independently K. Gregson (Thesis, University of Aberdeen, 1986) and K. Berman, H. Halpern, V. Kaftal, and G. Weiss (*Matrix norm inequalities and the relative Dixmier property*, Integral Equations Operator Theory **11** (1988)) have shown that the class of matrices with nonnegative entries is compressible. J. Bourgain and L. Tzafriri have recently shown that matrices with relatively small entries are compressible.

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