COVER-AVOIDANCE AND FRATTINI ARGUMENT

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ABSTRACT. The results presented here develop further aspects of the theory of Fitting functors in finite (solvable) groups. These ideas evolved in investigating whether a Fitting functor with the cover-avoidance property necessarily satisfies the Frattini argument. Our results might be seen as an indication for a positive answer.

1. INTRODUCTION

Recently the concept of a Fitting functor in finite (solvable) groups has been central to various investigations (cf. [1–4; 6–8]). A map f which assigns to each group G in $\mathcal{S}$, the class of all finite solvable groups, a nonempty set $f(G)$ of subgroups of G is called a *Fitting functor* if the following property is satisfied:

Whenever $\alpha: G \rightarrow H$ is a monomorphism with $\alpha(G) \trianglelefteq H$, then

$$\{\alpha(U) \mid U \in f(G)\} = \{\alpha(G) \cap V \mid V \in f(H)\}.$$

It has been shown in [1, 2] that Fitting functors satisfying the Frattini argument (that is, $G = N_G(U \cap N)N$ for all $G \in \mathcal{S}$, $N \trianglelefteq G$, $U \in f(G)$) appear to be the most interesting and manageable ones. They can be characterized in various different ways, most importantly as unions of conjugate Fitting functors. In this context, there is one intriguing question (already mentioned in [2]) whether Fitting functors with the cover-avoidance property (see 3.1) satisfy the Frattini argument.

This paper is mainly concerned with properties of these two types of Fitting functors. We content ourselves here to describe two of the results:

1. For quotient stable Fitting functors (that is $UN/N \in f(G/N)$ for all $G \in \mathcal{S}$, $N \trianglelefteq G$, $U \in f(G)$) the concepts of Frattini argument and cover-avoidance coincide. In fact, exactly the unions of Hall subgroup functors have these properties.

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2. Persistent Fitting functors (that is, $U \leq H \leq G$, $U \in f(G)$ implies $U \in f(H)$) satisfying the Frattini argument are just the unions of injector functors. We have not been able to decide whether the same result holds for persistent Fitting functors with the cover-avoidance property.

For unexplained notation used in this paper we refer the reader to [1, 2].

2. SPECIAL TYPES AND CONSTRUCTIONS OF FITTING FUNCTORS

We begin by recalling the basic definitions. For later purposes we separate the various properties that describe the concept of a Fitting functor.

2.1. Definition. Let $\mathcal{D}$ be a class of finite groups. (a) A *subgroup functor* f on domain $\mathcal{D}$ associates to each group $G \in \mathcal{D}$ a collection $f(G)$ of subgroups of G such that if $\alpha: G \rightarrow H$ is an isomorphism, then $f(H) = \{\alpha(X) \mid X \in f(G)\}$.

(b) *Condition (a)* is satisfied by a subgroup functor f on domain $\mathcal{D} = s_n\mathcal{D}$ provided that for each $G \in \mathcal{D}$, $N \trianglelefteq G$, and $U \in f(G)$, $U \cap N \in f(N)$.

(c) *Condition (b)* is satisfied by a subgroup functor f on domain $\mathcal{D} = s_n\mathcal{D}$ provided that for each $G \in \mathcal{D}$, $N \trianglelefteq G$, and $W \in f(N)$, there exists $U \in f(G)$ such that $U \cap N = W$.

(d) A *Fitting functor* on domain $\mathcal{D} = s_n\mathcal{D}$ is a subgroup functor on $\mathcal{D}$ that satisfies both conditions (a) and (b). In case $\mathcal{D} = \mathcal{S}$, we usually omit the reference to the domain.

We note in passing that Fitting functors are called *s_n -invariant subgroup embedding properties* by Hawkes [9]. Fitting functors were introduced in [1] to abstract the properties of radicals and injectors for Fitting classes. Various possibilities for constructions of Fitting functors were presented in [1]. Here we add another one which produces Fitting functors satisfying a strong form of condition (b). Since we take a closer look at this property later on, we introduce a special notation.

2.2. Definition. Let $\mathcal{D}$ be a class of finite groups, $\mathcal{D} = s_n\mathcal{D}$. *Condition (B^*)* is satisfied by a subgroup functor f on $\mathcal{D}$ provided for each $G \in \mathcal{D}$, $N \trianglelefteq G$, $f(N) \subseteq f(G)$.

2.3. Remark. It is clear that if f is a Fitting functor on $\mathcal{D}$ satisfying (B^*) , then $1 \in f(G)$ for all $G \in \mathcal{D}$. In particular, if $\mathcal{D} = \mathcal{S}$, then $\mathcal{L}_\pi(f) = \mathcal{S}_\pi$ for any set π of primes (cf. [1; 3.7] for the definition of $\mathcal{L}_\pi(f)$).

There is a rather general construction for Fitting functors satisfying condition (B^*) . To this end let f be a subgroup functor on $\mathcal{D} = s_n\mathcal{D}$ which satisfies condition (A). For $G \in \mathcal{S}$ define

$$f_0(G) = \{U \mid U \leq G, \text{ there exists } X \leq G, X \in \mathcal{D} \text{ such that } U \in f(X)\},$$

$$f_1(G) = \{U \mid U \leq G, \text{ there exists } X \trianglelefteq G, X \in \mathcal{D} \text{ such that } U \in f(X)\}.$$

2.4. Proposition. Under the assumptions above, f_0 and f_1 are Fitting functors satisfying (B^*) .

Proof. This is a straightforward application of the definitions.

2.5. Examples. (a) Let $\mathcal{D} = \mathcal{S}$, $f(G) = \{X \mid X \trianglelefteq G\}$ for $G \in \mathcal{S}$. Then f_0 and f_1 are just the examples f_s and f_{sn} in [1; 3.3.a].

(b) Let $\mathcal{D} = \{S_3, A_3, 1\}$, $f(S_3) = \text{Syl}_2(S_3)$, $f(A_3) = \{1\}$, $f(1) = \{1\}$. Certainly f satisfies condition (a). One readily checks that $f_0(G) = \{U \mid U \leq G, U = 1 \text{ or there exists } X \leq G, X \cong S_3, U \in \text{Syl}_2(X)\}$ for $G \in \mathcal{S}$ and similarly for f_1 (with $X \leq G$ replaced by $X \trianglelefteq G$).

This method is the only general one, other than the new-from-old constructions described in [1], that we have found for generating Fitting functors. However, this effort to find examples having the cover-avoidance property (see 3.1 for the exact definition) without satisfying the Frattini argument was in vain. In fact, we show in 3.4 that the only Fitting functor with the cover-avoidance property satisfying condition (B^*) is the trivial one ($f(G) = \{1\}$ for all $G \in \mathcal{S}$).

We conclude this section by showing that a certain condition on the class

$$\mathcal{H}(f) = \{H \mid \text{there exists } G \in \mathcal{S}, H_0 \in f(G), H_0 \cong H\}$$

(f a Fitting functor) forces f to satisfy condition (B^*) . However, as the example $f = f_s$ shows, this condition does not characterize Fitting functors satisfying condition (B^*) .

2.6. Proposition. *Let f be a Fitting functor such that for each $H \in \mathcal{H}(f)$ and each prime p dividing $|H/H'|$ there exists an elementary abelian p -group P such that $H \times P \notin \mathcal{H}(f)$. Then f satisfies condition (B^*) .*

Proof. Suppose the assertion is false. Then there exists a group G with maximal normal subgroup M , $|G : M| = p$, and $U \in f(M)$ such that $U \notin f(G)$.

To obtain a contradiction it suffices to show that $U \times C_p \in f(M \times C_p)$ and $U \times C_p \notin f(G \times C_p)$. For any $V \in f(G)$ such that $U \leq V$, $G = MV$ and so $|V : U| = p$. Fix an element $x \in G \setminus M$ such that $G = M\langle x \rangle$ and $U\langle x \rangle \in f(G)$. Choose n such that $p^n - 1 > |G|$ and let E be an elementary abelian group of order p^n . For each $e \in E$, $e \neq 1$, there is a homomorphism $\sigma_e : G \rightarrow E$ with $\sigma_e(g) = e^i$ for $g \in Mx^i$, $0 \leq i < p$. Thus there is an injection $\lambda_e : G \rightarrow G \times E$ given by $\lambda_e(g) = (g, \sigma_e(g))$. Let $G_e = \lambda_e(G) \cong G$. Clearly, $G_e \trianglelefteq G \times E$, $M \times 1 \trianglelefteq G_e$. Let $W \in f(G \times E)$ such that $W \cap (M \times 1) = U \times 1$. Then $W \cap G_e \in f(G_e)$. Then there exists $y_e \in G$ such that $(y_e, e) \in W \cap G_e$, for otherwise $U \times 1 = \lambda_e(U \times 1) \in f(G_e)$ in contradiction to $U \notin f(G)$. By the choice of n , there exist $e_1, e_2 \in E \setminus \{1\}$, $e_1 \neq e_2$, such that $y_{e_1} = y_{e_2}$. Then $(1, e_1^{-1}e_2) \in W \cap G_{e_1^{-1}e_2}$ whence $U \times \langle e_1^{-1}e_2 \rangle = W \cap (M \times \langle e_1^{-1}e_2 \rangle) \in f(M \times \langle e_1^{-1}e_2 \rangle)$, that is $U \times C_p \in f(M \times C_p)$. Clearly though $U \times C_p \notin f(G \times C_p)$ for otherwise $U \times 1 = (U \times C_p) \cap (G \times 1) \in f(G \times 1)$.

3. FITTING FUNCTORS SATISFYING THE COVER-AVOIDANCE PROPERTY

As mentioned in the introduction, the main purpose of this paper is the investigation of Fitting functors satisfying the cover-avoidance property. The

precise meaning of this concept is contained in the following definition.

3.1. Definition. Let f be a Fitting functor on domain $\mathcal{D} = s_n \mathcal{D}$. f has the *cover-avoidance property* provided for each $G \in \mathcal{D}$, each $U \in f(G)$ and each chief factor H/K of G , U either covers or avoids H/K (that is, either $H \leq KU$ or $H \cap U \leq K$).

It is easy to verify that a Fitting functor on domain $\mathcal{D} = s_n \mathcal{D} \subseteq \mathcal{S}$ satisfying the Frattini argument has the cover-avoidance property. The converse is not true for arbitrary domains $\mathcal{D}$, even if we restrict ourselves to Fitting classes $\mathcal{D} \subseteq \mathcal{S}$. This will be demonstrated in the following example. However, it is unknown whether for Fitting functors on domain $\mathcal{S}$, Frattini argument, and cover-avoidance property are equivalent. In the sequel of this paper we present some results that might support a conjecture on the equivalence.

3.2. Example. Let $\mathcal{D} = \mathcal{N}^k$ ($k \in \mathbb{N}$) be the class of all finite solvable groups of nilpotent length at most k . We define f on $\mathcal{D}$ by $f(G) = \{U \mid U \geq G_{\mathcal{N}^{k-1}}\}$, $G \in \mathcal{D}$.

(a) f is a Fitting functor on $\mathcal{D}$. If $N \trianglelefteq G \in \mathcal{D}$, $U \geq G_{\mathcal{N}^{k-1}}$ then clearly $U \cap N \geq G_{\mathcal{N}^{k-1}} \cap N = N_{\mathcal{N}^{k-1}}$. On the other hand, for $V \in f(N)$ set $U = VG_{\mathcal{N}^{k-1}}$. Then $U \in f(G)$ and, by Dedekind's identity, $U \cap N = V$.

(b) f has the cover-avoidance property on $\mathcal{D}$. Let H/K be a chief factor of $G \in \mathcal{D}$, $U \in f(G)$. Clearly, $H \cap G_{\mathcal{N}^{k-1}} \leq K$ or $H \leq G_{\mathcal{N}^{k-1}}$. If $H \leq G_{\mathcal{N}^{k-1}}$, then U covers H/K . In case $H \cap G_{\mathcal{N}^{k-1}} \leq K$, H/K is G -isomorphic to $HG_{\mathcal{N}^{k-1}}/KG_{\mathcal{N}^{k-1}}$ whence H/K has prime order as $G/G_{\mathcal{N}^{k-1}}$ is nilpotent. Thus H/K is covered or avoided by U by trivial reasons.

(c) f does not satisfy the Frattini argument. We use the fact that in a nilpotent group P a subgroup Q satisfies the Frattini argument (that is $P = M \cdot N_P(Q \cap M)$ for all normal subgroups M of P) if and only if Q is normal in P . Therefore it suffices to choose a group $G \in \mathcal{D}$ such that $G/G_{\mathcal{N}^{k-1}}$ is a (nilpotent) group possessing a nonnormal subgroup to exhibit an example demonstrating the above assertion.

It follows from [2; 3.1] in combination with [1; 7.2] that a Fitting functor f satisfying the Frattini argument satisfies the conclusion of the following theorem. We show here that actually the cover-avoidance property suffices.

3.3. Theorem. Let f be a Fitting functor (on $\mathcal{S}$) satisfying the cover-avoidance property. Let p be a prime number. Then the following hold:

- (a) If P is a p -group, then $f(P) \subseteq \{1, P\}$.
- (b) If $G \in \mathcal{S}$, $U \in f(G)$, and $p \mid |U|$, then $O_p(G) \leq U$.

Proof. (a) Let E be an elementary abelian p -group, $|E| = p^n$. Let Q be any irreducible subgroup of $\text{Aut}(E) \cong \text{GL}(n, p)$, $H = EQ$. Let $V \in f(E)$ and choose $U \in f(H)$ such that $U \cap E = V$. Now U is a cover-avoid subgroup of H and E is a chief factor of H . Hence $U \cap E = 1$ or $U \cap E = E$. This establishes the result in case $P = E$ is elementary abelian. The next step is to

show that for a Sylow p -subgroup P of $\text{GL}(n, p)$ the result is valid. We choose P as the group of lower triangular matrices with 1 on the main diagonal. For each i , $1 \leq i \leq n-1$, let $D_i = \{g = (g_{jk}) \in P \mid g_{jk} = 0 \text{ for } 1 \leq k < j \leq n-i \text{ and for } n-i+1 \leq k < j \leq n\}$. Then D_i is a normal elementary abelian subgroup of P , $P = \langle D_i \mid 1 \leq i \leq n-1 \rangle$, and $Z(P) \leq \bigcap_{i=1}^{n-1} D_i$. Let $W \in f(P)$. If $W \cap Z(P) = 1$, then $W \cap D_i = 1$ by the first part of the proof. Hence $W = 1$. If $W \cap Z(P) \neq 1$, then again by the first part $D_i \leq W$ for each i . Hence $W = P$. The final step of the proof is only to point out that every p -group is isomorphic to a subnormal subgroup of a Sylow p -subgroup of $\text{GL}(n, p)$ for some n . From this the conclusion follows.

(b) Suppose the assertion is false and let G be a minimal counterexample. Then there is $U \in f(G)$ such that $p \mid |U|$ but $O_p(G) \not\leq U$. Since $U \cap O_p(G) \in f(O_p(G))$, it follows from (a) that $U \cap O_p(G) = 1$.

Let M be a maximal normal subgroup of G such that $O_p(G) \leq M$. Then $U \cap M \in f(M)$. Since $O_p(G) = O_p(M)$, it follows from the minimality of G that $p \nmid |U \cap M|$. Thus $G = UM$ and $|G : M| = p$. Let p^k be the highest p -power dividing $|G|$. Let $H = EQ$ where E is an elementary abelian p -group of order greater than p^k , and Q is a p -complement in H such that Q acts irreducibly on E .

Let $\varphi : G \rightarrow H$ be a homomorphism with $\ker \varphi = M$. In $G \times H$, $G_0 = \{(g, \varphi(g)) \mid g \in G\}$ is a subnormal subgroup isomorphic to G (via the isomorphism $\sigma : G \rightarrow G_0$ given by $\sigma(g) = (g, \varphi(g))$). There exists $W \in f(G \times H)$ such that $W \cap G_0 = \sigma(U)$. Since $\sigma(U)$ does not avoid $G \times E/G \times 1$, W must cover $G \times E/G \times 1$. Thus $p^{k+1} \mid |W|$. Then, by order considerations, W cannot avoid E and hence covers E . But $E \leq O_p(G \times H)$ and so by (a), $O_p(G \times H) \leq W$. But $O_p(M) \times 1 \leq W \cap (M \times 1) = (U \cap M) \times 1$, a contradiction. The proof is complete.

3.4. Corollary. *Let f be a Fitting functor with the cover-avoidance property. If f satisfies condition (B^*) , then $f(G) = \{1\}$ for all groups $G \in \mathcal{S}$.*

Proof. Suppose there is a group G with $f(G) \neq \{1\}$. Choose $U \in f(G)$ with $U \neq 1$ and let p be a prime divisor of $|U|$. By condition (B^*) , $U \times 1 \in f(G \times C_p)$. This contradicts 3.3.

3.5. Example. Let $f(G) = \{U \mid U \leq G, U = 1 \text{ or there exists } X \trianglelefteq G, X \cong S_3, U \in \text{Syl}_2(X)\}$ for $G \in \mathcal{S}$.

By 2.5(b) and 2.4, f is a Fitting functor satisfying condition (B^*) . Therefore, by 3.4, f does not satisfy the cover-avoidance property. However, a weak form of the cover-avoidance property is valid for f . More precisely, for each group $G \in \mathcal{S}$ and each $U \in f(G)$ there exists a chief series of G such that all chief factors in the series are either covered or avoided by U .

One way to see this is to consider the normal Fitting class $\mathcal{F}$ defined by determinants on 3-chief factors [5; 3.3]. Then $|G/G_{\mathcal{F}}| = 1$ or 2 for all $G \in \mathcal{S}$. Suppose there exists $X \trianglelefteq G$, $X \cong S_3$. Let $U \in \text{Syl}_2(X)$. Since $X \notin \mathcal{F}$,

$G = G_{\mathcal{F}} \cdot U$, $U \cap G_{\mathcal{F}} = 1$. In any chief series of G passing through $G_{\mathcal{F}}$, U covers $G/G_{\mathcal{F}}$ and avoids all chief factors below $G_{\mathcal{F}}$.

4. QUOTIENT STABLE FITTING FUNCTORS SATISFYING THE COVER-AVOIDANCE PROPERTY

As mentioned earlier, we know of no example of a Fitting functor (on $\mathcal{S}$) with cover-avoidance property which does not satisfy the Frattini argument. On the other hand, a possible proof for the equivalence of these two concepts seems to be out of reach at the moment. In the present section we verify the equivalence for a special class of Fitting functors, the quotient stable ones.

4.1. Definition. A Fitting functor on $\mathcal{D} = s_n \mathcal{D} = Q\mathcal{D}$ is called *quotient stable* provided for each $G \in \mathcal{D}$, $N \trianglelefteq G$, and $U \in f(G)$, $UN/N \in f(G/N)$.

The following theorem actually classifies quotient stable Fitting functors with cover-avoidance property.

4.2. Theorem. Let f be a quotient stable Fitting functor satisfying the cover-avoidance property. Then there exists a set I and for each $i \in I$ a set π_i of primes such that $f = \bigcup_{i \in I} \text{Hall}_{\pi_i}$.

Proof. We show first that for each $G \in \mathcal{S}$ the collection $f(G)$ consists entirely of Hall subgroups of G . Suppose this is not the case and let G be a counterexample of minimal order. Let $U \in f(G)$ such that U is not a Hall subgroup of G . Clearly, $G \neq 1$; so let N be a minimal normal subgroup of G . Then $UN/N \in f(G/N)$ whence UN/N is a Hall subgroup of G/N . N is a p -subgroup of G for some prime p , and U covers or avoids N . If U avoids N , then, since U is not a Hall subgroup of G , $p \mid |U|$. But then by 3.3, $N \leq U$, a contradiction. Therefore U contains N , and the fact that U is not a Hall subgroup of G implies that $p \mid |G : U|$. Let H/K be a p -chief factor of G not covered by U . Consider $G \times C_p$. There exists $W \in f(G \times C_p)$ such that $W \cap (G \times 1) = U \times 1$. Thus $p \mid |W|$ and so $1 \times C_p \leq W$ by 3.3. Consequently, $W = U \times C_p$. Now $W(K \times 1)/K \times 1 \in f(G \times C_p/K \times 1)$, $p \mid |W(K \times 1)/K \times 1|$, and $H \times 1/K \times 1 \leq O_p(G \times C_p/K \times 1)$. Again by 3.3, $H \times 1/K \times 1 \leq W(K \times 1)/K \times 1$, that is, by virtue of $W = U \times C_p$, U covers H/K , a contradiction.

We have shown that $f(G)$ consists of Hall subgroups for each $G \in \mathcal{S}$. In particular, f satisfies the Frattini argument. According to [1; 7.2], $f = \bigcup_{i \in I} f_i$ for a suitable index set I where each f_i is a conjugate Fitting functor. Let $\pi_i = \{q \mid q \text{ is a prime and there exists } G \in \mathcal{S} \text{ and } X \in f_i(G) \text{ such that } q \mid |X|\}$.

To see that $f_i = \text{Hall}_{\pi_i}$, choose a group G such that for some $X \in f_i(G)$, $X \notin \text{Hall}_{\pi_i}(G)$. Let $p \in \pi_i$ such that $p \mid |G : X|$. Let $H \in \mathcal{S}$ such that $p \mid |Y|$ for $y \in f_i(H)$. Consider $T \in f_i(G \times H)$. By the conjugacy of f_i , we can suppose that $T \cap (G \times 1) = X \times 1$ and $T \cap (1 \times H) = 1 \times Y$. Clearly then, $p \mid |T|$ and $p \mid |G \times H : T|$. This contradicts the fact that $T \in f_i(G \times H)$ is a Hall subgroup of $G \times H$.

5. PERSISTENT FITTING FUNCTORS SATISFYING THE FRATTINI ARGUMENT

In this section we shall consider persistent Fitting functors, a notion introduced by Hawkes [9]. We recall the definition.

5.1. Definition. A Fitting functor f on domain $\mathcal{D} = s\mathcal{D}$ is called *persistent* provided that for each $G \in \mathcal{D}$, $U \in f(G)$, and $H \leq G$ with $U \leq H$, $U \in f(H)$.

Even for the special class of persistent Fitting functors we did not succeed in proving the equivalence of cover-avoidance property and Frattini argument. In [1; 3.10] it was shown that conjugate persistent Fitting functors are the functors derived from injectors for Fitting classes. In light of [1; 7.2] it seems natural to conjecture that persistent Fitting functors satisfying the Frattini argument are just unions of injector functors. This is in fact true, but the proof is not straightforward. We have to make use of the Lockett-construction for conjugate Fitting functors developed in [2] to establish this result.

5.2. Theorem. Let f be a persistent Fitting functor satisfying the Frattini argument. Then there exists a set I and, for each $i \in I$, a Fitting class $\mathcal{X}_i$ such that $f = \bigcup_{i \in I} \text{Inj}_{\mathcal{X}_i}$.

Proof. By [1; 7.2], $f = \bigcup_{i \in I} f_i$ where each f_i is a conjugate Fitting functor.

$$(1) \quad f^* = \bigcup_{i \in I} f_i^* \text{ is persistent.}$$

It follows from [2; 5.4] that $f^* = \bigcup_{i \in I} f_i^*$. Let $Y \in f^*(G)$, $Y \leq H \leq G$. There exists $Z \in f(G \times G)$ such that $\pi_i(Z) = Y$, $i = 1, 2$. Thus $Z \leq H \times H$ whence $Z \in f(H \times H)$. Therefore, $Y = \pi_1(Z) \in f^*(H)$.

$$(2) \quad \text{For each } i \in I, f_i^* = \text{Inj}_{\mathcal{Y}_i} \text{ for some Lockett class } \mathcal{Y}_i.$$

By [1; 3.10] it suffices to show that each f_i^* is persistent. Let $Y \in f_i^*(G)$, $Y \leq H \leq G$. Choose $R \in f_i^*(H)$. Then $T = Y \times R \in f_i^*(G \times H)$ by [2; 4.3]. Since $T \leq H \times H$, $T \in f^*(H \times H)$ by (1). Hence there exists some $j \in I$ such that $T \in f_j^*(H \times H)$. Consequently, $Y, R \in f_j^*(H)$. In particular, Y and R are conjugate in H . As $R \in f_i^*(H)$, it follows that $Y \in f_i^*(H)$.

$$(3) \quad \text{Let } G \in \mathcal{S} \text{ and } U \in f_i(G). \text{ Then there exists } U^* \in f_i^*(G) \text{ such that } U \leq U^* \text{ and } U \in f_i(U^*).$$

Let $Y \in f_i^*(G)$. Choose $T \in f_i(G \times Y)$ such that $\pi_1(T) = \pi_2(T) = Y$; this is possible by [2; 4.3] and the manner in which f_i^* is constructed. Let $\tilde{U} \times 1 = T \cap (G \times 1)$ and $1 \times R = T \cap (1 \times Y)$. Thus $\tilde{U} \in f_i(G)$, $R \in f_i(Y)$, and $\tilde{U}, R \trianglelefteq Y$. Since $T \leq Y \times Y$, $T \in f(Y \times Y)$, that is $T \in f_j(Y \times Y)$ for some $j \in I$. It follows that $\tilde{U}, R \in f_j(Y)$. Consequently, $\tilde{U}$ and R are conjugate in Y . Since $R \in f_i(Y)$ (as we have noted above), this implies that

$\tilde{U} \in f_i(Y)$. Clearly $\tilde{U}$ and U (both being members of $f_i(G)$) are conjugate in G , $\tilde{U}^g = U$ say. Now the assertion follows by putting $U^* = Y^g$.

(4) f_i is idempotent for each $i \in I$.

We prove the idempotency property by induction on $|G|$. Let $U \in f_i(G)$. Choose $U^* \in f_i^*(G)$ such that $U \in f_i(U^*)$ by (3). If $U^* < G$, then (as $U \in f_i(U^*)$) induction yields $U \in f_i(U)$. Otherwise $G = U^*$, $U \leq G$. Then $U = U \cap G \in f_i(U)$.

(5) $f_i = \text{Inj}_{\mathcal{L}(f_i)}$ for each $i \in I$.

Set $\mathcal{F}_i = \mathcal{L}(f_i)$. By (2), $f_i^* = \text{Inj}_{\mathcal{Y}_i}$ for some Lockett class $\mathcal{Y}_i$. Thus $\mathcal{Y}_i = \mathcal{L}(f_i^*) = \mathcal{F}_i^*$ where the latter equality follows from [2; 6.4]. Let $U \in f_i(G)$ and choose $U^* \in f_i^*(G) = \text{Inj}_{\mathcal{F}_i^*}(G)$ according to (3). We infer from [1; 3.2] that $N = (U^*)_{\mathcal{F}_i} \in \text{Inj}_{\mathcal{F}_i}(G)$. As $U \leq U^*$ and $U \in \mathcal{F}_i$ (by (4)), it follows that $U \leq N$. Moreover, $U \in f_i(U^*)$ by (3). Thus $U = U \cap N \in f_i(N) = \{N\}$, the latter equality because of $N \in \mathcal{F}_i = \mathcal{L}(f_i)$. Thus $U = N \in \text{Inj}_{\mathcal{F}_i}(G)$.

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