FILTRATIONS, ASYMPTOTIC AND PRÜFERIAN CLOSURES, CANCELLATION LAWS

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ABSTRACT. Let A be a commutative ring. For any filtration $f = (I_n)$ on the ring A let $\bar{f}$ (resp. $P(f)$, f') be the asymptotic (resp. prüferian, integral) closure of the filtration f . Then we have

$$(*) \quad f \leq f' \leq P(f) \leq \bar{f}.$$

In this paper several examples to show that each relation in $(*)$ may be an equality or a strict inequality even in noetherian rings, are given. Some transfer properties (such as the property that a filtration be AP or strongly AP) between the filtrations f , $P(f)$, and $\bar{f}$ are also given and negative answers are illustrated by some examples. This paper is closed by studying some cancellation laws concerning the prüferian closure of filtrations. In particular it is shown in the main theorem that if f , g , h are filtrations on the noetherian ring A such that $\sqrt{f} \subseteq \sqrt{h}$, if $P(fh) \leq P(gh)$ and if h is strongly AP then we have $P(f) \leq P(g)$. In this theorem the hypothesis " h strongly AP " cannot be weakened to " h AP " as shown in Example 2.3(3).

INTRODUCTION

In [7] result 2.3.2 and Theorem 2.6 (Cancellation law), the authors misquoted what Rees has shown in pages 11–12 of his paper entitled: *Asymptotic properties of ideals*, preprint. Indeed they did not require the filtrations f and g to be noetherian. Our paper originated with the attempt to show that Theorem 2.6 (Cancellation law) of [7] holds for arbitrary filtrations f and g provided that the filtration h be strongly AP . In addition we show here that this Cancellation law does not hold for arbitrary h even if h is AP . All rings are assumed to be commutative with an identity. I , J , ... etc. will denote arbitrary ideals of the given ring A . For any filtration $f = (I_n)$ on the ring A , the asymptotic (resp. prüferian, integral) closure $\bar{f}$ (resp. $P(f)$, f') of f are defined in §1.1 (resp. 1.2) below. We have the following inequalities for a given filtration f

$$(*) \quad f \leq f' \leq P(f) \leq \bar{f}.$$

In §1 of this paper we give some well-known relations between filtrations and

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pseudo-valuations that are needed later and we recall a cancellation law given by [8] concerning the asymptotic closure of filtrations.

In §2 we give various examples to show that the inequality between any two members in $(*)$ may be an equality or a strict inequality even in noetherian rings. We show also that the prüferian closure does not satisfy in general the same type of cancellation law as the asymptotic closure even if the filtration h satisfies nice properties such as the AP condition (see Example 2.3(3)).

We close this section with some transfer properties between f , $P(f)$, and $\bar{f}$. Although $P(f)$ and $\bar{f}$ are different in general, $P(f)$ is, however, very close to $\bar{f}$ (see 2.4(5)). We use this remark to show that $\bar{f}$ is AP if and only if $P(f)$ is AP (Proposition 2.5). However this equivalence is not true if we take “strongly AP ” in place of “ AP ” as shown in Example 2.4(6).

In §3 we give a weak cancellation law concerning the prüferian closure of filtrations. Then we prove in the main theorem (see Theorem 3.3) that if f, g, h are filtrations on the noetherian ring A such that $\sqrt{f} \subseteq \sqrt{h}$, if $P(fh) \leq P(gh)$ and if h is strongly AP , we have $P(f) \leq P(g)$.

In this theorem the hypothesis “ h strongly AP ” cannot be weakened to “ h AP ” as shown in Example 2.3(3).

1. ASYMPTOTIC CLOSURE OF FILTRATIONS

1.1. Definitions. (1) A filtration on the ring A is a sequence $f = (I_n)$ of ideals of A such that $I_0 = A$, $I_{n+1} \subseteq I_n$, and $I_n \cdot I_m \subseteq I_{n+m}$ for all $n, m \geq 0$. The set $\mathbb{F}(A)$ of all filtrations on the ring A is ordered by $f = (I_n) \leq g = (J_n)$ if $I_n \subseteq J_n$ for any two elements f, g in $\mathbb{F}(A)$. The product of the filtrations $f = (I_n)$ and $g = (J_n)$ is the filtration $fg = (I_n J_n)$.

(2) A pseudovaluation on the ring A is a mapping $v: A \rightarrow \mathbb{R}_+ \cup (\infty)$ such that

- (i) $v(0) = \infty$; $v(1) = 0$
- (ii) $v(x - y) \geq \min(v(x), v(y))$
- (iii) $v(xy) \geq v(x) + v(y)$, for all $x, y \in A$;

here $\mathbb{R}_+$ is the set of nonnegative real numbers.

(3) Let v be a pseudovaluation on A . The homogeneous pseudovaluation associated with v is the pseudovaluation $\bar{v}$ defined by $\bar{v}(x) = \lim_{n \rightarrow \infty} v(x^n)/n$ [4, Proposition 11.1]. We have $\bar{v} \geq v$.

(4) For any pseudovaluation v on A , we put $[v](x) = [v(x)]$ for all $x \in A$, where $[v(x)]$ is the greatest integer less than or equal to the real number $v(x)$. $[v]$ is a pseudovaluation on A that is integer-valued.

(5) We have $[\bar{v}] = \bar{v}$.

(6) With any filtration $f = (I_n) \in \mathbb{F}(A)$ we associate the pseudovaluation v_f such that $v_f(x) = \sup\{n; x \in I_n\}$ for all $x \in A$ (see [1] for generalizations of this concept).

For any ideal I of A we put $v_I = v_{f_I}$, where $f_I = (I^n)$ is the I -adic filtration; v_I is called the I -adic pseudovaluation.

(7) Conversely for any pseudovaluation v on A we put $f_v = (I_{v,n})_{n \geq 0}$, where $I_{v,n} = \{x \in A; v(x) \geq n\}$. f_v is a filtration on A and we have:

$$(8) \quad v_{f_v} = [v]; \quad f_{v_{f_v}} = f.$$

(9) The asymptotic closure of the filtration f is the filtration $\bar{f}$ associated with the homogeneous pseudovaluation $\bar{v}_f$; $\bar{f} = f_{\bar{v}_f}$. We have:

$$(10) \quad v_{\bar{f}} = v_{f_w} = [\bar{v}_f], \text{ where } w = \bar{v}_f. \text{ This follows from (8). We have also:}$$

$$(11) \quad \bar{v}_{\bar{f}} = [\bar{v}_f] = \bar{v}_f = \bar{v}_f. \text{ These are consequences of (5) and (10).}$$

The following theorem is a part of Theorem 2.10 of [8].

1.2. Theorem. *The operation $f \rightarrow \bar{f}$ is a semiprime operation on the set $\mathbb{F}(A)$. That is, for all filtrations f and g on A we have:*

- (i) $f \leq \bar{f}$;
- (ii) $\bar{\bar{f}} = \bar{f}$;
- (iii) if $f \leq g$, then $\bar{f} \leq \bar{g}$;
- (iv) $\bar{f}\bar{g} \leq \bar{f\bar{g}}$.

Furthermore we have the following cancellation law:

Let $f, g, h \in \mathbb{F}(A)$ such that $\sqrt{f} \subseteq \sqrt{h}$, if $\bar{f}h \leq \bar{g}h$ then $\bar{f} \leq \bar{g}$.

2. PRÜFERIAN CLOSURE OF FILTRATIONS

2.1. Definitions. (1) An element $x \in A$ is said to be integral over the filtration $f = (I_n)$ if x satisfies an equation $x^m + a_1x^{m-1} + \dots + a_jx^{m-j} + \dots + a_m = 0$, where $a_j \in I_j$ for all j .

(2) Let I be an ideal of A . An element $x \in A$ which is integral over the I -adic filtration f_I is said to be integral over I . We put $I' = \{x \in A; x \text{ is integral over } I\}$. I' is an ideal of A and is called the integral closure of the ideal I . This is a well-known result.

(3) Let k be an integer ≥ 1 . For any filtration $f = (I_n)$, $f^{(k)}$ denotes the filtration $(I_{nk})_{n \geq 0}$.

(4) Let $f = (I_n) \in \mathbb{F}(A)$ and let $k \geq 1$ be an integer. We put $P_k(f) = \{x \in A; x \text{ is integral over } f^{(k)}\}$. $P_k(f)$ is the set of elements $x \in A$ such that $x^m + a_1x^{m-1} + \dots + a_jx^{m-j} + \dots + a_m = 0$, for some integer $m \geq 1$, where $a_j \in I_{jk}$ for all j .

(5) The filtration $P(f) = (P_n(f))$ is called the prüferian closure of f following [3], or the weak integral closure of f following [7] and the filtration $f' = (I'_n)$ is called the integral closure of f .

(6) In particular $P_1(f)$ is the set of elements $x \in A$ which are integral over $f^{(1)} = f$ and $P_1(f_I) = I'$ for any ideal I of A .

Let $f = (I_n)$ and $g = (J_n)$ be two filtrations on A .

(7) g is said to be integral over f if $g \leq P(f)$, that is, $J_n \subseteq P_n(f)$ for all $n \geq 1$.

(8) Now we will give a list of well-known filtrations that behave like I -adic filtrations and we will order them according to a diagram. Let $f = (I_n) \in \mathbb{F}(A)$.

(a) f is said to be I -good, where I is an ideal of A , if $I \cdot I_n \subseteq I_{n+1}$ for all n and $I \cdot I_n = I_{n+1}$ for all large n .

(b) f is approximatable by powers and is called AP for short in case there exists a sequence of positive integers (k_n) such that $I_{k_n m} \subseteq I_n^m$ for all n, m and $k_n/n \rightarrow 1$ as $n \rightarrow \infty$.

(c) f is called strongly AP if there exists an integer $k \geq 1$ such that $I_{nk} = I_k^n$ for all n , that is $f^{(k)} = f_{I_k}$.

(d) f is said to be noetherian if the Rees ring $\mathcal{R}(A, f) = A[u, I_1 X, I_2 X^2, \dots]$, where X is an indeterminate and $u = X^{-1}$ is noetherian. This terminology is that introduced by D. Rees [10]. It has a different meaning in [2].

(e) f is strongly noetherian if A is noetherian and if there exists an integer $k \geq 1$ such that $I_m \cdot I_n = I_{m+n}$ for all $m, n \geq k$ (see [6]). We have in any ring A :

$$fI\text{-adic} \Rightarrow fI\text{-good} \Rightarrow f \text{ strongly } AP \Rightarrow fAP.$$

If A is noetherian then $f = (I_n)$ is noetherian if and only if there exists an integer $k \geq 1$ such that $I_{n+k} = I_n I_k$ for all $n \geq k$. Furthermore each noetherian filtration on a noetherian ring is strongly AP , (see [9, 2]). So in a noetherian ring A we have the following diagram for any filtration f :

$$\begin{array}{ccccc} fI\text{-adic} & \Rightarrow & fI\text{-good} & \Rightarrow & f \text{ strongly } AP & \Rightarrow & fAP \\ & & \downarrow & & \uparrow & & \\ & & f \text{ strongly} & \Rightarrow & f \text{ noetherian} & & \\ & & \text{noetherian} & & & & \end{array}$$

2.2. Relations between $f, f', P(f)$, and $\bar{f}$. We have

- (1) $f \leq f' \leq P(f) \leq \bar{f}$;
- (2) $P(f') = (P(f))' = P(f)$;
- (3) $\bar{f} = \overline{f'} = \overline{P(f)}$.

Proof. (1) is easy to show.

(2) We have $P(f) \leq P(f') \leq P(P(f)) = P(f)$ and $P(f) \leq P(f)' \leq P(P(f)) = P(f)$.

(3) follows from (1) and Theorem 1.2. $\square$

Each inequality in (1) may be an equality or a strict inequality. We have:

(4) For any ideal I , $(f_I)' = f_I$ if and only if I is normal (see [4, Chapter 11; 11]).

(5) $(f_I)' = P(f_I)$, as easily shown. Let $\bar{v}_I$ be the I -adic homogeneous pseudovaluation on the noetherian ring A (see 1.1(3) and 1.1(6)) and let k be an integer ≥ 1 . Then for each $x \in A$, we have:

(6) $\bar{v}_I(x) \geq k \Leftrightarrow x \in (I^k)'$. In other words, $P(f_I) = \bar{f}_I$. For a generalization of (6) see [1, Theorem 5.6]. Another generalization of (6) has been given by [3, Proposition 4.7] as follows.

(7) If A is noetherian and if the filtration f is strongly AP then $P(f) = \bar{f}$.

2.3. Examples. Let $A = k[X]$, where k is a field and X an indeterminate

(1) Let p be an integer ≥ 2 and let $f = (I_n)$ where $I_0 = A$,

$$I_n = \begin{cases} (X^n) & \text{if } n \geq p+1, \\ (X^p) & \text{if } 1 \leq n \leq p. \end{cases}$$

It is easy to show that f is a filtration which is strongly noetherian hence strongly AP ; $Y = X$ is integral over f since $Y^p - X^p = 0$ with $X^p \in I_p$. But $Y = X$ is not integral over $I_1 = (X^p)$. Hence $I'_1 \neq P_1(f)$ and $f' \neq P(f)$.

(2) Let $f = (I_n)$ where $I_0 = A$ and $I_n = (X^{n+1})$ for $n \geq 1$. f is a filtration on A which is AP (take $k_n = n+1$ for all n). We have $\bar{v}_f(X) = 1$, hence $X \in I_{\bar{v}_f, 1}$. But it is easy to show that $X \notin P_1(f)$, therefore $P(f) \neq \bar{f}$.

(3) Let $I = (X)$; $I_n = \begin{cases} I^n & \text{if } n \text{ is even,} \\ I^{n+1} & \text{if } n \text{ is odd,} \end{cases}$ $J_0 = A$; $J_n = I^{n+1}$ if $n \geq 1$, and let $f = (I_n)$, $g = h = (J_n)$. f, g, h are filtrations on A , f is noetherian, $g = h$ is AP . We have $\sqrt{f} = \sqrt{g} = \sqrt{h} = I$. If we put $fh = (K_n)$; $gh = (L_n)$, then $K_{2n} = I^{4n+1}$; $K_{2n-1} = I^{4n}$; $L_n = I^{2n+2}$ for all $n \geq 1$.

To show that $P(fh) \leq P(gh)$ it is enough to have $fh \leq P(gh)$, i.e., $K_n \subseteq P_n(gh)$ for all n . We have $K_{2n-1} = I^{4n} = L_{2n-1} \subseteq P_{2n-1}(gh)$; $K_{2n} = I^{4n+1}$, $L_{4n} = I^{8n+2}$. If we take $Y = X^{4n+1}$ then $Y^2 - X^{8n+2} = 0$, with $X^{8n+2} \in L_{4n}$, therefore $Y \in P_{2n}(gh)$, hence $K_{2n} \subseteq P_{2n}(gh)$ and $P(fh) \leq P(gh)$. But it is not true that $P(f) \leq P(g)$. For example $Y = X$ satisfies $Y^2 - X^2 = 0$, with $X^2 \in I_2$, that means $X \in P_1(f)$. But it is easy to show that $X \notin P_1(g)$, therefore $P_1(f)$ is not included in $P_1(g)$. This last example shows that the Cancellation law given in our Theorem 3.3 does not hold for arbitrary filtrations f, g, h even if h is AP .

2.4. Transfer properties. For any filtration f on the noetherian ring A the following implications are easily shown.

(1) $P(f)$ strongly $AP \Rightarrow f'$ strongly $AP \Rightarrow f$ strongly AP . Furthermore if one of the above conditions is fulfilled, then we have:

(2) $\bar{f} = P(f)$ [3, Proposition 4.7]. Following [10, Introduction], for any filtration f on the ring A there exists a pseudovaluation $(v_f)^*$ on A which satisfies:

(3) For $x \in A$ and $n \in \mathbb{N}$, $(v_f)^*(x) \geq n \Leftrightarrow x \in P_n(f)$. Therefore $I_{(v_f)^*, n} = P_n(f)$ for all n , hence $f_{(v_f)^*} = P(f)$ and $(v_f)^* = v_{f_{(v_f)^*}} = v_{P(f)}$. Following the same reference as above we have for $x \in A$:

(4) $(v_f)^*(x) \leq \bar{v}_f(x) < (v_f)^*(x) + 1$. An immediate consequence of (4) is

(5) $P_{n+1}(f) \subseteq I_{\bar{v}_f, n+1} \subseteq P_n(f)$, for all n . Relation (5) means that $P(f)$ is very close to $\bar{f}$. If in addition the filtration f is noetherian then, as suggested by the referee, we have $[\bar{v}_f(x)] = (v_f)^*(x)$ (see e.g. [10]), hence $\bar{f} = P(f)$.

Let us remark that with the general hypothesis $\bar{f} = P(f)$ and $\bar{g} = P(g)$, Theorem 2.6 of [7] is a consequence of Theorem 2.10 of [8] which is our The-

orem 1.2. Indeed if $P(fh) = P(gh)$ then $\overline{fh} = \overline{P(fh)} = \overline{P(gh)} = \overline{gh}$. If in addition $\sqrt{f} + \sqrt{g} \subseteq \sqrt{h}$ then by Theorem 1.2 $P(f) = \overline{f} = \overline{g} = P(g)$.

We may ask whether the diagram (1) can be completed by the implication $\overline{f}$ strongly $AP \Rightarrow P(f)$ strongly AP . This would imply that $\overline{f}$ strongly $AP \Rightarrow f$ strongly AP . This last implication is not true as shown by the following

(6) *Example.* Let A be a principal ideal domain and let p be an irreducible element of A . Consider the filtration $f = (I_n)$, where $I_0 = A$, $I_n = P^{n+1}$ for $n \geq 1$, with $P = pA$. We have $\overline{v}_f(p) = 1$, therefore, $\overline{v}_f(p^n) = n$ for all n . Since $I_{\overline{v}_{f,n}} \subseteq P \subseteq I_{\overline{v}_{f,n+1}}$, we have $P^n = I_{\overline{v}_{f,n}}$ for all n . It follows that $\overline{f} = f_P$ is P -adic hence strongly AP . But f is not strongly AP .

However, we have

2.5. Proposition. *Let $f \in \mathbb{F}(A)$. Then $\overline{f}$ is AP if and only if $P(f)$ is AP .*

Proof. Suppose that $\overline{f} = (J_n)$ is AP . Then there exists a sequence (k_n) of integers such that $\lim_{n \rightarrow \infty} k_n/n = 1$ and $J_{k_n m} \subseteq J_n^m$ for all n, m . By (5) we have $P_{k_n m}(f) \subseteq J_{k_n m} \subseteq J_n^m \subseteq (P_{n-1}(f))^m$. Therefore $P_{k_{n+1} m}(f) \subseteq (P_n(f))^m$, with $\lim_{n \rightarrow \infty} k_{n+1}/n = 1$ and $P(f)$ is AP . Conversely suppose that $P(f)$ is AP . Let (k_n) be a sequence of integers such that $P_{k_n m}(f) \subseteq (P_n(f))^m$ for all n, m and $\lim_{n \rightarrow \infty} k_n/n = 1$. We have for all $m \geq 1$, $J_{(k_n+1)m} \subseteq J_{k_n m+1} \subseteq P_{k_n m} \subseteq (P_n(f))^m \subseteq J_n^m$ by (5). So, $\overline{f}$ is AP since $\lim_{n \rightarrow \infty} (k_n + 1)/n = 1$. $\square$

2.6. To close this section, we know [3, Proposition 4.7] that for any strongly AP filtration f on the noetherian ring A , we have $P(f) = \overline{f}$. But the converse implication is not true, that is there exist filtrations f that are not strongly AP (even not AP) such that $P(f) = \overline{f}$ as shown in the following example. This fact is not surprising because for any filtration f , $P(f)$ is very close to $\overline{f}$.

Example. Let I be a maximal ideal of the ring A such that $I^2 \neq I$ and let $f = (I_n)$ where $I_0 = A$, $I_n = I$ for all $n \geq 1$. f is a filtration which satisfies $f = f' = P(f) = \overline{f}$. Indeed, by 2.2(1) we need only to show that $\overline{f} \leq f$, which is easily done. However f is not AP .

3. CANCELLATION LAWS FOR THE PRÜFERIAN CLOSURE OF FILTRATIONS

In this section we will give first a weak cancellation law for the prüferian closure (Corollary 3.2) and secondly a cancellation law which is like that verified by the asymptotic closure (Theorem 3.3). The next proposition is an adaptation of [5, Theorem 3].

3.1. Proposition. *Let $f = (I_n)$, $g = (J_n)$, and $h = (K_n)$ be filtrations on A . Suppose that K_n is finitely generated for all n . If $\sqrt{f} \subseteq \sqrt{h}$ and if $fh \leq gh$, then $f \leq g'$.*

Proof. We need only to show that $I_n \subseteq J'_n$ for all n , where J'_n is the integral closure of the ideal J_n . Suppose that K_n is generated by $c_1, c_2, \dots, c_r$. Let

$x \in I_n$. Then $c_i x = \sum_{j=1}^r b_{ij} c_j$, where $b_{ij} \in J_n$ for $i = 1, 2, \dots, r$. Hence $\sum_{j=1}^r (\delta_{ij} x - b_{ij}) c_j = 0$, where $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $j \neq i$. Consider the matrix $M = (\delta_{ij} x - b_{ij})$. We have $MX = 0$, where ${}^tX = (c_1, \dots, c_r)$. Let d be the determinant of M . Then it is well known that there exists a matrix $\tilde{M}$ such that ${}^t\tilde{M}MX = dX = 0$, where ${}^t\tilde{M}$ is the transpose of $\tilde{M}$. It follows that $dc_i = 0$ for all i and $dK_n = 0$. We have $dx \in \sqrt{dI_n} = \sqrt{dA} \cap \sqrt{f} \subseteq \sqrt{dA} \cap \sqrt{h} = \sqrt{dK_n} = \sqrt{0}$. So there exists an integer $s \geq 1$ such that $d^s x^s = 0$. Finally $d = x^r + a_1 x^{r-1} + \dots + a_j x^{r-j} + \dots + a_r$, where $a_j \in J_n^j$ for $j = 1, 2, \dots, r$; $d^s = x^{rs} + b_1 x^{rs-1} + \dots + b_j x^{rs-j} + \dots + b_{rs}$, with $b_j \in J_n^j$ for $j = 1, 2, \dots, r$. It follows that $0 = d^s x^s = x^{rs+s} + b_1 x^{rs} + b_1 x^{rs+s-1} + \dots + b_{rs} x^s$, hence $x \in J_n'$. $\square$

3.2. Corollary. Let f, g, h be filtrations on the noetherian ring A such that $\sqrt{f} + \sqrt{g} \subseteq \sqrt{h}$. If $P(f)P(h) = P(g)P(h)$, then $P(f) = P(g)$.

Proof. We have $\sqrt{P(f)} = \sqrt{f} \subseteq \sqrt{h}$ and $(P(f))' \leq (P(g))'$ by Proposition 3.1. Hence by 2.2(2) $P(f) \leq P(g)$. The conclusion follows by symmetry. $\square$

3.3. Theorem. Let f, g, h be filtrations on a noetherian ring A such that $\sqrt{f} \subseteq \sqrt{h}$. If h is strongly AP and if $P(fh) \leq P(gh)$, then $P(f) \leq P(g)$.

Proof. Suppose first that f and g are strongly AP . Since the product of two strongly AP filtrations is strongly AP the conclusion follows by Theorem 1.2 and [3, Proposition 4.7].

Now suppose that $f = (I_n)$ and $g = (J_n)$ are arbitrary filtrations. For all integers $m \geq 1$, let f_m be the smallest filtration whose first $m+1$ terms are $A, I_1, I_2, \dots, I_m$. f_m is noetherian [2, Proposition 2.8]. Hence f_m is strongly AP [9, Remark 2.4.3]. We have $fh \leq P(gh)$ which implies $f_m h \leq P(gh)$ since $f_m \leq f$. Let $f_m h = (K_n)$; $f_m h$ is strongly AP . There exists an integer $r \geq 1$ such that $(f_m h)^{(r)} = f_{K_r}$. Moreover $f_m h$ is integral over $gh = (L_n)$, hence K_r is integral over $(gh)^{(r)}$. Suppose that K_r is generated by $x_1, x_2, \dots, x_t$, then there exists an integer $N \geq 1$ such that:

$$(*) \quad x_i^N + a_{i,1} x_i^{N-1} + \dots + a_{i,j} x_i^{N-j} + \dots + a_{i,N} = 0, \text{ where } a_{ij} \in L_{jr} \\ \text{for all } j = 1, 2, \dots, N \text{ and } i = 1, 2, \dots, t.$$

Let g_{Nr} be the smallest filtration whose first $Nr+1$ terms are $A, J_1, J_2, \dots, J_{Nr}$. As shown above g_{Nr} is noetherian, hence strongly AP . By (*), K_r is integral over $(g_{Nr})^{(r)}$, which implies that $f_m h$ is integral over $g_{Nr} h$ by [3, Corollary 4.3]. Therefore $P(f_m h) \leq P(g_{Nr} h)$. As $\sqrt{f_m} = \sqrt{f} \subseteq \sqrt{h}$, then it follows from the first part of this proof that $f_m \leq P(f_m) \leq P(g_{Nr}) \leq P(g)$ for all $m \geq 1$. Hence $I_m \subseteq P_m(g)$ for all m and $P(f) \leq P(g)$. $\square$

3.4. Corollary. Let f, g, h be filtrations on the noetherian ring A such that $\sqrt{f} + \sqrt{g} \subseteq \sqrt{h}$. If h is strongly AP and if $P(fh) = P(gh)$, then $P(f) = P(g)$.

Proof. This corollary is an immediate consequence of Theorem 3.3. $\square$

3.5. *Remark.* As shown in Example 2.3(3), the Cancellation law 3.3 does not hold with the weaker condition “ h AP.”

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