MINIMAL IMMERSIONS OF PUNCTURED COMPACT RIEMANN SURFACES IN $\mathbb{R}^3$

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ABSTRACT. We prove that a hyperelliptic Riemann surface of genus g can be completely conformally and minimally immersed in $\mathbb{R}^3$ with finite total curvature with at most $3g+2$ punctures; an arbitrary compact Riemann surface of genus g can be so immersed with at most $4g$ punctures. Moreover, we show that there is at least a one-parameter family of nonisometric such immersions for a given punctured compact Riemann surface. Our results improve earlier results of Gackstatter-Kunert and the author.

1. INTRODUCTION AND THE STATEMENT OF THE RESULT

Let M be a connected Riemann surface and consider a conformal minimal immersion $f: M \rightarrow \mathbb{R}^3$. It is a fundamental theorem due to Chern and Osserman that for a complete f (i.e., the induced metric on M is complete) the total curvature is finite if and only if the Gauss map is algebraic. For the sake of simplicity let us call a complete conformal minimal immersion $f: M \rightarrow \mathbb{R}^3$ of finite total curvature an *algebraic minimal surface*. (We warn the reader that there are other notions of algebraic minimal surfaces.) In particular, if $f: M \rightarrow \mathbb{R}^3$ is an algebraic minimal surface, then M is, via a biholomorphism, identified with a compact Riemann surface M_g punctured at finitely many points, and the Gauss map of f extends holomorphically to all of M_g .

Klotz and Sario [KS] proved that there exists an algebraic minimal surface of every genus. Hoffman and Meeks [HM] later constructed an algebraic minimal surface of every genus with exactly three punctures that is actually embedded. On the other hand Gackstatter and Kunert proved the following immersion theorem:

Theorem A [GK]. *Any compact Riemann surface of genus g can be immersed in $\mathbb{R}^3$ as an algebraic minimal surface with at most $4g+1$ punctures.*

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Later the author proved the following result:

Theorem B [Y1]. *Given any nonconstant meromorphic function F_1 on a compact Riemann surface M_g of genus $g > 0$, there exists another meromorphic function F_2 on M_g such that $\{dF_1, F_2\}$ is the Weierstrass pair defining a complete conformal minimal immersion of finite total curvature into $\mathbb{R}^3$ defined on M_g punctured at the supports of the polar divisors of F_1 and F_2 .*

Since there is always an abundant supply of meromorphic functions on a Riemann surface, the above theorem implies that any compact Riemann surface can be immersed in $\mathbb{R}^3$ as an algebraic minimal surface with finitely many punctures. Indeed in [Y1] the author using Theorem B recovered the Gackstatter-Kunert theorem and proved the following result.

Theorem C [Y1]. *Any hyperelliptic Riemann surface of genus g can be immersed in $\mathbb{R}^3$ as an algebraic minimal surface with at most $3g + 4$ punctures.*

In the present paper we will improve Theorems A-C by establishing the following result:

Theorem. *Let M_g be any compact Riemann surface of genus $g > 0$. Then there exists at least a one-parameter family of nonisometric algebraic minimal surfaces $M_g \setminus S \rightarrow \mathbb{R}^3$, where S is a finite set. For $g = 1$ we can have S with $|S| \leq 5$. For $g \geq 2$ and M_g hyperelliptic we can have S with $|S| \leq 3g + 2$. For $g \geq 2$ and M_g arbitrary we can have $|S| \leq 4g$.*

2. THE WEIERSTRASS REPRESENTATION

In this section we briefly review the Weierstrass representation of a minimal surface in $\mathbb{R}^3$; we set up notations to be used in the next section. For proofs and details the reader may consult [Y2].

Consider a conformal map $f = (f^i): M \rightarrow \mathbb{R}^3$, where M is a Riemann surface. Conformality means that the induced metric $f^*ds_E^2$, ds_E^2 the Euclidean metric, is in the conformal class of M given by its complex structure. We know that f is minimal if and only if $\Delta f \equiv 0$, where Δ is the Laplacian of $(M, f^*ds_E^2)$. If z is a local holomorphic coordinate on M , then for some $h(z) > 0$ we have $f^*ds_E^2 = h(z) dz \cdot d\bar{z}$, and we compute that $\Delta = -\frac{4}{h} \cdot \frac{\partial^2}{\partial z \partial \bar{z}}$.

Suppose f is minimal, and put $\eta^i = \partial f^i / \partial z$. Then $\partial \eta^i / \partial \bar{z} = 0$, i.e., each η^i is a local holomorphic function on M . In fact, it is easy to verify that the 1-forms $\zeta^i = \eta^i dz$ are globally defined holomorphic 1-forms on M . The assignment $z \mapsto [\eta^1(z), \eta^2(z), \eta^3(z)]$ gives a well-defined holomorphic map $\Phi_f: M \rightarrow \mathbb{CP}^2$ which is the holomorphic Gauss map. We note that $f^i(z) = 2 \operatorname{Re} \int^z \zeta^i$.

Reversing the above process we can manufacture minimal surfaces from holomorphic 1-forms.

Proposition. Let M be a Riemann surface, and suppose we have holomorphic 1-forms $\zeta^1, \zeta^2, \zeta^3$ on M satisfying the following conditions:

- (a) $\sum |\eta^i|^2 > 0$, where $\zeta^i = \eta^i dz$ locally;
- (b) $\sum (\eta^i)^2 = 0$;
- (c) the ζ^i 's have no real periods.

Then the formula $f_\zeta = (f^i) = 2 \operatorname{Re} \int_{z_0}^z (\zeta^i): M \rightarrow \mathbb{R}^3$ defines a conformal minimal immersion with $f(z_0) = 0$.

Remark. Even without condition (c) the forms (ζ^i) would define a conformal minimal immersion on the universal cover of M .

Let M be a Riemann surface, and suppose we are given a nonconstant meromorphic function φ and a not identically zero holomorphic 1-form μ on M . We further suppose that φ has a pole of order m at $z \in M$ if and only if μ has a zero of order $2m$ at z . Put

$$\zeta^1 = \frac{1}{2}(1 - \varphi^2)\mu, \quad \zeta^2 = \frac{i}{2}(1 + \varphi^2)\mu, \quad \zeta^3 = \varphi\mu.$$

It is then easily verified that the forms (ζ^i) define a conformal minimal immersion

$$f_\zeta: M \rightarrow \mathbb{R}^3$$

given that they have no real periods. Up to congruence every minimal surface in $\mathbb{R}^3$ arises in this manner, and we call $\{\mu, \varphi\}$ the Weierstrass pair representing f_ζ .

We note that

$$\varphi = \pi \circ \Phi_f^\perp: M \rightarrow S^2 \rightarrow \mathbb{C} \cup \{\infty\} = \mathbb{CP}^1,$$

where $\Phi_f^\perp$ is the normal Gauss map and π is the stereographic projection. We record that

$$f^* ds_E^2 = h(z) dz \cdot d\bar{z}, \quad h(z) = |\eta|^2 (1 + |\varphi|^2)^2,$$

where $\mu = \eta dz$ locally.

Let $f: M \rightarrow \mathbb{R}^3$ be a conformal minimal immersion, and also let K denote the Gaussian curvature of the induced metric. Then $K \leq 0$, and

$$\tau_f = \int_M K dA \leq 0$$

is called the total curvature of f . The Gauss map of f , Φ_f , is said to be algebraic if the following holds:

- (1) M is biholomorphic to a compact Riemann surface M_g punctured at a finite set of points $\{p_1, \dots, p_r\}$;
- (2) Φ_f extends holomorphically to all of M_g .

A theorem of Chern-Osserman states that a complete conformal minimal immersion f is of finite total curvature if and only if its Gauss map is algebraic.

3. THE PROOF

In this section we prove Theorem stated in §1. The present proof is a refinement of the proof of Theorem B given in [Y1].

Let M_g be a compact Riemann surface of genus $g > 0$, and also let F_1 be any nonconstant meromorphic function on M_g . For some $b_i \in \mathbb{Z}^+$ and distinct points $p_i \in M_g$ we have $(F_1)_\infty = \sum_{i=1}^n b_i p_i$. Put $d = \sum b_i = \deg(F_1)_\infty$. Consider the meromorphic 1-form dF_1 . We have $(dF_1)_\infty = \sum (b_i + 1)p_i$. For some $a_j \in \mathbb{Z}^+$ and distinct points $q_j \in M_g$ we have $(dF_1)_0 = \sum_{j=1}^m a_j q_j$. Since $(dF_1) = (dF_1)_0 - (dF_1)_\infty$ is a canonical divisor we must have $\deg(dF_1) = 2g - 2$ so that $\sum a_j = (2g - 2) + n + d$. Introduce a divisor $D \in \text{Div}(M_g)$ by $D = \sum_{j=1}^m a_j q_j - \sum_{i=1}^n c_i p_i = D^+ - D^-$, where the c_i 's are some positive integers satisfying the conditions $c_i \geq b_i + 1$; $\sum c_i = 3g - 2 + n + d$. The first condition means that $D^- \geq (dF_1)_\infty$, and the second condition means that $\deg D = -g$. Consider the complex vector space $L(-D) = \{F \in H^0(M_g, \mathcal{M}^*) : (F) \geq D\} \cup \{0\}$. $\mathcal{M}^*$ denotes the sheaf of germs of not identically zero meromorphic functions on M_g . Hence a not identically zero meromorphic function F is a member of $L(-D)$ if and only if it has zeros of order at least a_j at q_j and poles of order at most c_i at p_i (or no poles). Now $\deg(dF_1) = 2g - 2 \geq 0$ and since $(dF_1)_\infty > 0$ we must have

$$\text{support}(dF_1)_0 \neq \emptyset.$$

Consequently nonzero constant functions cannot be in $L(-D)$. By the Riemann-Roch theorem

$$\begin{aligned} \dim L(-D) &= \deg(-D) - g + 1 + \dim L((dF_1) + D) \\ &= 1 + \dim L((dF_1) + D) \geq 1. \end{aligned}$$

So we can (and do) choose a nonconstant meromorphic function $G \in L(-D)$.

Remark. We may replace G by a nonzero complex multiple, and obtain a one-parameter family of noncongruent minimal surfaces in $\mathbb{R}^3$, however, we will not exploit this fact in this paper.

Returning to the proof we put

$$(G)_0 = \sum_{j=1}^m \tilde{a}_j q_j + \sum_{k=1}^l \tilde{a}_{m+k} q_{m+k}; \quad (G)_\infty = \sum_{i=1}^n \tilde{c}_i p_i$$

with

$$\tilde{c}_i \leq c_i, \quad \tilde{a}_j \geq a_j; \quad \sum \tilde{a}_j + \sum \tilde{a}_{m+k} = \sum \tilde{c}_i.$$

The two inequalities mean that $G \in L(-D)$, and the equality comes from the fact that (G) is a principal divisor. Define a nonconstant meromorphic function F_2 on M_g by

$$F_2 = \sum_{\alpha=1}^{\lambda} c_\alpha / G^\alpha,$$

where $\lambda = 2(n + m + l - 1) + (4g + 1)$, and (c_α) is a nonzero vector in $\mathbb{C}^\lambda$ to be chosen suitably later. Consider meromorphic 1-forms $F_2 dF_1$ and $F_2^2 dF_1$ on

M_g . Observe that

$$\{q_{m+1}, \dots, q_{m+l}\} \subset \text{support}(F_2 dF_1)_\infty \subset \{q_1, \dots, q_{m+l}; p_1, \dots, p_n\},$$

$$\{q_1, \dots, q_{m+l}\} \subset \text{support}(F_2^2 dF_1)_\infty \subset \{q_1, \dots, q_{m+l}; p_1, \dots, p_n\}.$$

Claim. We can choose $(c_\alpha) \in \mathbb{C}^\lambda \setminus \{0\}$ such that the forms $F_2 dF_1$ and $F_2^2 dF_1$ have neither residues nor periods on M_g .

Put

$$R_{i,\alpha} = \text{Res}_{p_i} dF_1 / G^\alpha, \quad R_{j,\alpha} = \text{Res}_{q_j} dF_1 / G^\alpha, \quad R_{k,\alpha} = \text{Res}_{q_{m+k}} dF_1 / G^\alpha,$$

where Res denotes the residue. We then have

$$\begin{aligned} \text{Res}_{p_i} F_2 dF_1 &= \sum_\alpha c_\alpha R_{i,\alpha}, \\ \text{Res}_{q_j} F_2 dF_1 &= \sum_\alpha c_\alpha R_{j,\alpha}, \\ \text{Res}_{q_{m+k}} F_2 dF_1 &= \sum_\alpha c_\alpha R_{k,\alpha}. \end{aligned}$$

So the meromorphic 1-form $F_2 dF_1$ on M_g has no residues if and only if the c_α 's satisfy the following system of linear equations:

$$(I) \quad \sum c_\alpha R_{i,\alpha} = 0; \quad \sum c_\alpha R_{j,\alpha} = 0; \quad \sum c_\alpha R_{k,\alpha} = 0.$$

The system has $n + m + l$ equations. We know that on a compact Riemann surface the total residue of any meromorphic 1-form must vanish. It follows that if any $n + m + l - 1$ residues of $F_2 dF_1$ were to vanish, then the remaining residue would also have to vanish. Consequently, we may (and do) throw out an equation from the linear system (I). Let $(e_1, \dots, e_{2g})$ be 1-cycles on M_g representing a canonical homology basis. Put

$$P_{a,\alpha} = \int_{e_a} dF_1 / G^\alpha, \quad 1 \leq a \leq 2g, 1 \leq \alpha \leq \lambda.$$

So the e_a -period of $F_2 dF_1$ is given by the sum $\sum c_\alpha P_{a,\alpha}$. It follows that the differential $F_2 dF_1$ has no periods on M_g if and only if

$$(II) \quad \sum c_\alpha P_{a,\alpha} = 0, \quad 1 \leq a \leq 2g.$$

This system is linear in (c_α) and contains at most $2g$ independent equations. We now consider the differential $F_2^2 dF_1$. Put

$$R_{i,2\alpha} = \text{Res}_{p_i} dF_1 / G^{2\alpha}, \quad 1 \leq \alpha \leq \lambda.$$

Similarly define $R_{j,2\alpha}$, and $R_{k,2\alpha}$. Thus the residue of $F_2^2 dF_1$ at p_i is given by

$$R_i(c_\alpha) = \sum_{\alpha=1}^{\lambda} c_\alpha^2 R_{i,2\alpha} + 2 \sum_{1 \leq \alpha < \beta \leq \lambda} c_\alpha c_\beta R_{i,\alpha+\beta}.$$

We also let $R_j(c_\alpha)$ and $R_k(c_\alpha)$ denote the residues of $F_2^2 dF_1$ at q_j and q_{m+k} respectively. Observe that $R_i(c_\alpha)$, $R_j(c_\alpha)$, and $R_k(c_\alpha)$ are all homogeneous polynomials in (c_α) of degree 2. The zero locus in $\mathbb{C}^\lambda$ of one of these polynomials is a (possibly degenerate) homogeneous quadratic. Now the meromorphic differential $F_2^2 dF_1$ has no residues if and only if the c_α 's satisfy

$$(III) \quad R_i(c_\alpha) = 0; \quad R_j(c_\alpha) = 0; \quad R_k(c_\alpha) = 0.$$

Again using the vanishing of the total residue we can throw out one of the equations from (III). Put

$$P_{a,2\alpha} = \int_{e_a} dF_1 / G^{2\alpha}, \quad P_{a,\alpha+\beta} = \int_{e_a} dF_1 / G^{\alpha+\beta},$$

where $1 \leq \alpha, \beta \leq \lambda$, $1 \leq a \leq 2g$. So the e_a -period of $F_2^2 dF_1$ is given by

$$P_a(c_\alpha) = \sum c_\alpha^2 P_{a,2\alpha} + 2 \sum_{1 \leq \alpha < \beta \leq \lambda} c_\alpha c_\beta P_{a,\alpha+\beta}.$$

The form $F_2^2 dF_1$ has no periods on M_g if and only if

$$(IV) \quad P_a(c_\alpha) = 0, \quad 1 \leq a \leq 2g.$$

Each $P_a(c_\alpha)$ is a homogeneous polynomial in (c_α) of degree 2. The number of equations in the system (I-IV) is $2(n+m+l-1) + 4g = \lambda - 1$.

Proof of Claim. Note that the solution set of an equation in (I-IV) is either a hyperplane or a (possibly degenerate) homogeneous quadric in $\mathbb{C}^\lambda$. At any rate it corresponds to an algebraic hypersurface in $\mathbb{P}^{\lambda-1}$. But we know that $\lambda - 1$ algebraic hypersurfaces in $\mathbb{P}^{\lambda-1}$ must intersect. (This follows at once from the codimension formula: for any two algebraic varieties $V_1, V_2 \subset \mathbb{P}^N$,

$$\text{codim}(V_1 \cap V_2) \leq \text{codim } V_1 + \text{codim } V_2.)$$

In fact we see that the set of c_α 's solving (I-IV) is itself a homogeneous affine variety. In particular, if a vector (c_α) solves the system, then so does any complex multiple of it. We let $\text{Sol} \subset \mathbb{C}^\lambda$ denote this solution variety. Fix $c = (c_\alpha) \in \text{Sol} \setminus \{0\}$. We define holomorphic 1-forms $\zeta^1, \zeta^2, \zeta^3$ on

$$M = M_g \setminus \{\text{support}(F_1)_\infty \cup \text{support}(F_2)_\infty\} = M_g \setminus \{p_i; q_j; q_{m+k}\}$$

by the formulae

$$\zeta^1 = \frac{1}{2}(1 - F_2^2)dF_1, \quad \zeta^2 = \frac{i}{2}(1 + F_2^2)dF_1, \quad \zeta^3 = F_2 dF_1.$$

The fact that the differentials $F_2 dF_1$ and $F_2^2 dF_1$ have no residues and no periods on M_g guarantees that the holomorphic differentials $\zeta^1, \zeta^2, \zeta^3$ have no real periods on M . Consequently the formula

$$f = (f^\varepsilon) = 2 \operatorname{Re} \int^z (\zeta^\varepsilon), \quad 1 \leq \varepsilon \leq 3,$$

defines a conformal minimal immersion $M \rightarrow \mathbb{R}^3$. At a puncture

$$p \in \{p_i; q_j; q_{m+k}\} = M_g \setminus M$$

each ζ^ε has at worst a pole, hence the Gauss map of f extends holomorphically to all of M_g (cf. [Y2, p. 29]). It is not hard to see that any path approaching one of the punctures has an infinite arclength. Take a p_i , for example. If we let z be a holomorphic coordinate centered at p_i and write

$$\zeta^\varepsilon = \eta^\varepsilon dz,$$

then

$$h(z) = 2 \sum |\eta^\varepsilon|^2 = C/|z|^{2m} + \text{higher order terms}, \quad m \geq 2$$

since dF_1 has a pole of order $b_i + 1 \geq 2$ at p_i . The arclength of any path approaching p_i must be infinite since the induced metric on M is given by

$$f^* ds_E^2 = h(z) dz \cdot d\bar{z}$$

near the puncture. Thus the induced metric is complete. We have shown that each

$$c = (c_\alpha) \in \text{Sol} \setminus \{0\}$$

gives rise to a complete conformal minimal immersion of finite total curvature

$$f_c: M = M_g \setminus S \rightarrow \mathbb{R}^3,$$

where S denotes the finite puncture set. Let $\tilde{c} = (\tilde{c}_\alpha) \in \text{Sol} \setminus \{0\}$ be given by

$$\tilde{c}_\alpha = \rho c_\alpha,$$

where ρ is any nonconstant complex number. Let $\{d\tilde{F}_1 = dF_1, \tilde{F}_2\}$ be the Weierstrass pair coming from the choice $\tilde{c} \in \text{Sol} \setminus \{0\}$. We see that

$$\tilde{F}_2 = \rho F_2.$$

Let $\tilde{h}(z) dz \cdot d\bar{z}$ denote a local expression for the induced metric of $\tilde{f} = f_{\tilde{c}}$. We compute that $\tilde{h} = |\eta|^2(1 + |\rho F_2|^2)^2$, where $dF_1 = \eta dz$ locally. On the other hand, the induced metric of f_c is given by $h(z) dz \cdot d\bar{z}$ with $h = |\eta|^2(1 + |F_2|^2)^2$. It follows that the surfaces f_c and $f_{\tilde{c}}$ are not isometric for $|\rho| \neq 1$ showing that there exists at least a one-parameter family of nonisometric complete conformal minimal immersions of finite total curvature $M_g \setminus S \rightarrow \mathbb{R}^3$. Let $\#$ denote the number of punctures, i.e., $\# = |S|$. So $\# = n + m + l$, where

$$n = \text{the number of distinct poles of } F_1,$$

$$m = \text{the number of distinct zeros of } dF_1.$$

Moreover, $m + l \leq (3g - 2) + d + n$, where $d = \deg(F_1)_\infty$. This is so since $m + l \leq \sum \tilde{c}_i \leq \sum c_i = 3g - 2 + d + n$. Thus $\# \leq (3g - 2) + d + 2n$. Suppose M_g is a complex torus, i.e., $g = 1$. Then for any point $p \in M_g$ we can

find a meromorphic function F on M_g such that $(F)_\infty = 2p$. Set $F_1 = F$. Then $d = 2$ and $n = 1$. Hence $\sharp \leq 5$. Suppose M_g is hyperelliptic, and let $p \in M_g$ be a Weierstrass point. We then know that there exists a meromorphic function F on M_g with $(F)_\infty = 2p$. Let $F_1 = F$ we see that $d = 2$, $n = 1$, and $\sharp \leq 3g + 2$. We now suppose that M_g is an arbitrary Riemann surface of genus $g \geq 2$. On M_g there are at least $2g + 2$ Weierstrass points. Let $p \in M_g$ be a Weierstrass point. This means that the gap sequence at p is not given by $\{1, 2, \dots, g\}$. Since there are exactly g gaps it follows that we must have a nongap $d \leq g$. (The worst possible gap sequence at p is $\{1, 2, \dots, g - 1, g + 1\}$.) But this means that there is a meromorphic function F on M_g with $(F)_\infty = dp$. Letting $F_1 = F$ we have $n = 1$, and $\sharp \leq (3g - 2) + (d + 2n) \leq (3g - 2) + (g + 2) \leq 4g$. $\square$

Our proof suggests that the bounds on the number of punctures given in Theorem may in fact be sharp. It would be interesting to verify this.

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