ON INTERMEDIATE DIFFERENTIABILITY OF LIPSCHITZ FUNCTIONS ON CERTAIN BANACH SPACES

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ABSTRACT. A real-valued function f defined on a Banach space X is said to be intermediately differentiable at $x \in X$ if there is $\xi \in X^*$ such that for every $h \in X$ the value $\langle \xi, h \rangle$ lies between the upper and lower derivatives of f at x in the direction h . We show that if Y contains a dense continuous linear image of an Asplund space and X is a subspace of Y , then every locally Lipschitz function on X is generically intermediately differentiable.

Let $(X, \|\cdot\|)$ be a Banach space with dual X^* and duality pairing between X^* and X denoted by $\langle \cdot, \cdot \rangle$. Recall that the *upper* and *lower derivatives* of a function $f: X \rightarrow \mathbb{R}$ at $x \in X$ in a direction $h \in X$ are defined by

$$D^+ f(x, h) = \limsup_{t \downarrow 0} \frac{1}{t} [f(x + th) - f(x)]$$

and

$$D_+ f(x, h) = \liminf_{t \downarrow 0} \frac{1}{t} [f(x, th) - f(x)]$$

respectively. The function f is said to be *intermediately differentiable* at $x \in X$, with *intermediate derivative* $\xi \in X^*$, if

$$D^+ f(x, h) \geq \langle \xi, h \rangle \geq D_+(x, h) \quad \text{for all } h \in X.$$

The aim of this note is to prove the following statement.

Theorem. *Suppose that a Banach space Y contains a dense continuous linear image of an Asplund space and that X is a subspace of Y .*

Then every locally Lipschitz function defined on an open subset Ω of X is intermediately differentiable at every point of a residual subset of Ω .

The Banach spaces containing a dense continuous linear image of an Asplund space have been extensively studied by Ch. Stegall [11], who calls them *GSG spaces*. Among other things, he also proved [11, §4, Remark 3] that the non-w.c.g. subspace of some $L_1(\mu)$, μ finite, constructed by H. Rosenthal [10] is

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not GSG. Hence our theorem extends the result of M. Fabian and N. V. Zhivkov [5] who considered the case $X = Y$.

We observe that our result implies the well-known fact [2] that a subspace X of a GSG space is weak Asplund. Indeed, using [15, Proposition 2.1], one easily sees that a stronger statement holds: If a continuous function f defined on an open subset Ω of X has (one-sided) directional derivative at each point of Ω and at each direction, then f is generically Gateaux differentiable on Ω .

It does not seem that the proof of the above theorem can be obtained by adapting the proofs from [5, 15]. In fact, a quite different approach works; we use some ideas met in the theory of analytic spaces on one hand and some two-dimensional reasoning similar to those used in [8, 9] on other hand.

The crucial concept we use is d - A -dentability*. Let E be a bounded subset of the dual X^* of a Banach space X , let A be a bounded absolutely convex subset of X ; and let $d > 0$ be given. We say that E is d - A -dentable* if there are $0 \neq e \in X$ and $\gamma > 0$ such that the slice

$$S(E, e, \gamma) \equiv \{\xi \in E : \langle \xi, e \rangle > \sup \langle E, e \rangle - \gamma\}$$

has A -diameter less than d ; that is,

$$\text{diam}_A S(E, e, \gamma) \equiv \sup\{\|\xi - \eta\|_A : \xi, \eta \in S(E, e, \gamma), u \in A\} < d.$$

Let us recall that $A \subset X$ is said to have a *Stegall property* provided that every bounded subset of X^* is d - A -dentable* for every $d > 0$; see [1, pp. 116–148], where other equivalent characterizations and other names of this property can also be found.

We also use the notion of a *Clarke subdifferential* [3]: Given a locally Lipschitz function $f : X \rightarrow \mathbb{R}$ on a Banach space X , put

$$f^0(x)(h) = \limsup_{\substack{t \downarrow 0 \\ y \rightarrow x}} \frac{1}{t} [f(y + th) - f(y)], \quad x, h \in X.$$

Then the Clarke subdifferential of f at $x \in X$ is defined by

$$\partial f(x) = \{\xi \in X^* : \langle \xi, h \rangle \leq f^0(x)(h) \text{ for all } h \in X\}.$$

Note that $\partial f(x)$ is a nonempty set and that the function $f^0(x)(\cdot)$ is subadditive and positively homogeneous for each $x \in X$.

The key lemma we use can be formulated as:

Lemma. Let G be an open subset of a Banach space X , consider a Lipschitz function $f : G \rightarrow \mathbb{R}$ with a Lipschitz constant 1, and put $\partial f(G) = \bigcup \{\partial f(x) : x \in G\}$. Let A be an absolutely convex bounded subset of X with the Minkowski's functional q . Finally, let $d > 0$, $\gamma > 0$, and $e \in X$, with $\|e\| = 1$, be such that

$$(1) \quad \text{diam}_A S(\partial f(G), e, \gamma) < d.$$

Then there exist x in G , ξ in the unit ball B_{X^*} of X^* and $\Delta > 0$ such that

$$|f(x+h) - f(x) - \langle \xi, h \rangle| \leq 3dq(h)$$

whenever $h \in X$ and $q(h) < \Delta$.

Proof. Put $\alpha = \sup\{\|u\| : u \in A\}$ and

$$(2) \quad s = \sup\langle \partial f(G), e \rangle$$

Choose

$$(3) \quad \beta \in \left(0, \min\left(\gamma, \frac{d^2\gamma}{4\alpha(d+2\alpha)}\right)\right).$$

From the definition of $\partial f(G)$ and using classical facts from real analysis, we can find $x \in G$ such that the limit

$$Vf(x)(e) \equiv \lim_{t \rightarrow 0} \frac{1}{t} [f(x+te) - f(x)]$$

exists and $Vf(x)(e) > s - \beta$. In fact, find $y \in G$ such that $s - \beta < \sup\langle \partial f(y), e \rangle$; then, according to the definition of $\partial f(y)$, we get $s - \beta < f^0(y)(e)$. Also, by the definition of $f^0(y)(e)$, find $z \in G$ and $\tau > 0$ so that $\{z + te : t \in [0, \tau]\} \subset G$ and $s - \beta < \frac{1}{\tau} [f(z + \tau e) - f(z)]$. Now the mapping $t \mapsto g(t) \equiv f(z + te)$, $t \in [0, \tau]$, is Lipschitz. Hence the derivative $g'(t)$ exists almost everywhere and moreover,

$$\int_0^\tau g'(t) dt = f(z + \tau e) - f(z),$$

which is greater than $\tau(s - \beta)$. It follows that there is $t \in (0, \tau)$ such that $g'(t)$ exists and $g'(t) > s - \beta$. Then put $x = z + te$.

Further, from the definition of $\partial f(x)$, we can find $\xi \in \partial f(x)$ such that

$$(4) \quad Vf(x)(e) = \langle \xi, e \rangle \quad (> s - \beta).$$

Note that $\xi \in S(\partial f(G), e, \gamma)$ since $\beta < \gamma$. Also there is $\delta \in (0, 1)$ such that $u \in G$ whenever $\|u - x\| < 2\delta$ and such that

$$(5) \quad |f(x + \tau e) - f(x) - \tau \langle \xi, e \rangle| \leq \beta |\tau| \quad \text{for } \tau \in (-\delta, \delta).$$

Now fix any $h \in X$ such that $0 < q(h) < \min(\delta/\alpha, 2\beta\delta/d)$, and take

$$(6) \quad t = dq(h)/2\beta.$$

Then note that $\|h\| \leq \alpha q(h) < \delta$ and $0 < t < (d/2\beta) \cdot (2\beta\delta/d) = \delta$. Let us remark that for $(\tau, \rho) \in [0, t] \times [0, 1] \equiv I$ we have

$$\|x + \tau e + \rho h - x\| \leq \tau + \rho \|h\| \leq t + \|h\| < 2\delta;$$

so $x + \tau e + \rho h \in G$. Thus for $\rho \in [0, 1]$, we get

$$\begin{aligned} f(x + \tau e + \rho h) &\geq f(x + \tau e) - \rho \|h\| \\ &\geq f(x) + t \langle \xi, e \rangle - \beta t - \rho \|h\| \\ &> f(x) + ts - 2\beta t - \|h\| \end{aligned}$$

by (5) and (4); and further

$$f(x + \rho h) \leq f(x) + \|h\|.$$

And putting the last two inequalities together with $\|h\| \leq \alpha q(h)$, (6), and (3), we get

$$\begin{aligned} f(x + te + \rho h) - f(x + \rho h) &> ts - 2\beta t - 2\|h\| \\ &\geq ts - 2\beta t - 2\alpha q(h) \\ &= ts - 2\beta t(1 + 2\alpha/d) > ts - d\gamma t/2\alpha, \end{aligned}$$

(7) $f(x + te + \rho h) - f(x + \rho h) > ts - d\gamma t/(2\alpha) \quad \text{for all } \rho \in [0, 1].$

We now define a mapping $\phi : I \rightarrow \mathbb{R}$ by

$$\phi(\tau, \rho) = f(x + \tau e + \rho h), \quad (\tau, \rho) \in I.$$

Note that ϕ is Lipschitz and so by the Rademacher theorem, there is a Lebesgue negligible set N in I such that the Gateaux (i.e. Fréchet) derivative $\phi'(\tau, \rho)$ exists whenever $(\tau, \rho) \in I \setminus N$. Let us observe that for these (τ, ρ)

$$\begin{aligned} \phi'_1(\tau, \rho) &= Vf(x + \tau e + \rho h)(e) \leq s, \\ \phi'_2(\tau, \rho) &= Vf(x + \tau e + \rho h)(h) \geq -\|h\| \geq -\alpha q(h), \end{aligned}$$

where ϕ'_1 and ϕ'_2 mean the partial derivatives of ϕ along the first and the second coordinate respectively. Let λ_1 and λ_2 denote the one- and the two-dimensional Lebesgue measures respectively.

Consider the following sets:

$$\begin{aligned} (8) \quad E &= \{(\tau, \rho) \in I \setminus N : \phi'_1(\tau, \rho) > s - \gamma\}; \\ E_\tau &= \{\rho \in [0, 1] : (\tau, \rho) \in E\}, \quad \tau \in [0, t]; \\ E^\rho &= \{\tau \in [0, t] : (\tau, \rho) \in E\}, \quad \rho \in [0, 1]. \end{aligned}$$

Using (7) and (8) we can estimate for almost all $\rho \in [0, 1]$,

$$\begin{aligned} ts - \frac{d\gamma t}{2\alpha} &< \phi(t, \rho) - \phi(0, \rho) = \int_0^t \phi'_1(\tau, \rho) d\tau = \int_{E^\rho} + \int_{E^{\rho c}} \\ &\leq \lambda_1(E^\rho)s + (t - \lambda_1(E^\rho))(s - \gamma) = \lambda_1(E^\rho)\gamma + ts - t\gamma; \\ \lambda_1(E^\rho) &> t - dt/(2\alpha); \\ \Lambda_2(e) &= \int_0^1 \lambda_1(E^\rho) d\rho > t - dt/(2\alpha). \end{aligned}$$

On the other hand, as $\lambda_2(E) = \int_0^t \lambda_1(E_\tau) d\tau$, we can find $\tau \in [0, t]$ such that

$$(9) \quad \lambda_1(E_\tau) > 1 - d/(2\alpha)$$

and that, simultaneously, the derivative $\phi'(\tau, \rho)$ exists for almost all $\rho \in [0, 1]$. For every such ρ we have $(\tau, \rho) \notin N$ and by the Hahn-Banach theorem, there exists $\eta_\rho \in \partial f(x + \tau e + \rho h)$ such that

$$\langle \eta_\rho, u \rangle = Vf(x + \tau e + \rho h)(u)$$

for all u from the linear span of e and h . Therefore, observing that, by (8), $\eta_\rho \in S(\partial f(G), e, \gamma)$ for $\rho \in E_\tau$, we have by (1) and (9)

$$\begin{aligned} |f(x + \tau e + h) - f(x + \tau e) - \langle \xi, h \rangle| &= \left| \int_0^1 \phi'_2(\tau, \rho) d\rho - \langle \xi, h \rangle \right| \\ &= \left| \int_0^1 \langle \eta_\rho - \xi, h \rangle d\rho \right| \leq \left| \int_{E_\tau} \right| + \left| \int_{E_\tau^c} \right| \\ &\leq \lambda_1(E_\tau) dq(h) + (1 - \lambda_1(E_\tau)) 2\|h\| \\ &\leq dq(h) + 2d\|h\|/(2\alpha) \leq 2dq(h), \end{aligned}$$

$$(10) \quad |f(x + \tau e + h) - f(x + \tau e) - \langle \xi, h \rangle| \leq 2dq(h).$$

Finally, note that $x + \tau e, x + \tau e + h \in G$. So we can estimate

$$\begin{aligned} (11) \quad f(x + \tau e) &\geq f(x) + \tau \langle \xi, e \rangle - \beta \tau \\ &\geq f(x) + \tau(s - \beta) - \beta \tau \geq f(x) + \tau s - 2\beta t \end{aligned}$$

by (5) and (4). Also, there are $\rho_n \uparrow 1$ such that

$$\begin{aligned} f(x + \tau e + \rho_n h) &= \phi(\tau, \rho_n) = \phi(0, \rho_n) + \int_0^\tau \phi'_1(\theta, \rho_n) d\theta \\ &= f(x + \rho_n h) + \int_0^\tau V f(x + \theta e + \rho_n h)(e) d\theta \\ &\leq f(x + \rho_n h) + \tau s, \end{aligned}$$

hence

$$f(x + \tau e + h) \leq f(x + h) + \tau s.$$

Now by combining the last inequality with (11), we obtain

$$f(x + h) - f(x) > f(x + \tau e + h) - f(x + \tau e) - 2\beta t.$$

Thus, putting this inequality together with (10) and (6), we get

$$f(x + h) - f(x) - \langle \xi, h \rangle > -2dq(h) - dq(h) = -3dq(h).$$

So far we have been working in the parallelepiped with vertices $x, x + te, x + h$, and $x + te + h$. Performing an analogous reasoning in the parallelepiped with vertices $x, x - te, x + h$, and $x - te + h$ we can find a $\tilde{\tau} \in (0, t)$ such that

$$|f(x - \tilde{\tau} e + h) - f(x - \tilde{\tau} e) - \langle \xi, h \rangle| \leq 2dq(h)$$

and

$$f(x + h) - f(x) < f(x - \tilde{\tau} e + h) - f(x - \tilde{\tau} e) + 2\beta t.$$

By combining the last two inequalities and (6) we get

$$f(x + h) - f(x) - \langle \xi, h \rangle < 3dq(h).$$

Proof of Theorem. Let Z an Asplund space and $T : Z \rightarrow Y$ be a linear continuous mapping such that $T(Z)$ is dense in Y . We work in the subspace X

of Y . Let Ω be an open subset of X and $f : \Omega \rightarrow \mathbb{R}$ be a locally Lipschitz function. According to [6, Chapter 1, §10] we know that a set $M \subset \Omega$ is residual in Ω if every point $x \in \Omega$ has a neighborhood W such that $M \cap W$ is residual in W . It follows that we may assume without loss of generality, that f is (globally) Lipschitz on the whole Ω . Also, we may and do assume that the Lipschitz constant of f is equal to 1.

For $n, p \in \mathbb{N}$ put

$$A_{n,p} = [nT(B_Z) + \frac{1}{p}B_Y] \cap X,$$

where B_Z, B_Y denote the unit balls in Z, Y respectively. Then we claim that every subset in the unit ball B_{X^*} of X^* is $3/p - A_{n,p}$ -dentable*. Assuming this fact, then the lemma says that the sets

$$M_{n,p} = \left\{ x \in \Omega : \exists \xi \in B_{X^*} \exists \Delta > 0 |f(x+h) - f(x) - \langle \xi, h \rangle| \leq (9/p)q_{n,p}(h) \right. \\ \left. \text{whenever } h \in X \text{ and } q_{n,p}(h) < \Delta \right\}$$

are dense in Ω , where $q_{n,p}$ are the Minkowski's functionals of $A_{n,p}$. Put

$$\widetilde{M}_{n,p} = \{x \in \Omega : \exists \xi \in B_{X^*} \exists r \in (0, 1/p) \\ \sup\{|f(x+rh) - f(x) - \langle \xi, rh \rangle| : h \in A_{n,p}\} < 10r/p\}.$$

These sets are open because f is Lipschitz. They are also dense in Ω since they contain $M_{n,p}$. Thus putting $M = \bigcap_{n,p=1}^{\infty} \widetilde{M}_{n,p}$, this set is residual in Ω .

We show that M is the set, which we are looking for. So fix some $x \in M$. For any sequence $\sigma = (n_1, n_2, \dots) \in \mathbb{N}^{\mathbb{N}}$ put $A_{\sigma} = \bigcap_{p=1}^{\infty} A_{n_p,p}$. Let us fix for a moment one such σ . Then for each $p \in \mathbb{N}$ we can find $r_p \in (0, 1/p)$ and $\xi_p \in B_{X^*}$ such that

$$\left| \frac{1}{r_p} [f(x + r_p h) - f(x)] - \langle \xi_p, h \rangle \right| < \frac{10}{p} \quad \text{whenever } h \in A_{n_p,p}.$$

As the sequence $\{\xi_p\}$ lies in B_{X^*} , it has a weak* cluster point ξ_{σ} , say. Fix for a moment any $h \in A_{\sigma}$. For each $i = 1, 2, \dots$ there is $p_i > i$ such that $|\langle \xi_{p_i} - \xi_{\sigma}, h \rangle| < 1/i$. Then

$$\left| \frac{1}{r_{p_i}} [f(x + r_{p_i} h) - f(x)] - \langle \xi_{\sigma}, h \rangle \right| \\ \leq \left| \frac{1}{r_{p_i}} [f(x + r_{p_i} h) - f(x)] - \langle \xi_{p_i}, h \rangle \right| + |\langle \xi_{p_i} - \xi_{\sigma}, h \rangle| \\ < 10/p_i + 1/i < 11/i \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

And, recalling that $0 < r_{p_i} < 1/p_i < 1/i$, we can conclude that

$$(12) \quad D^+ f(x, h) \geq \langle \xi_{\sigma}, h \rangle \geq D_+ f(x, h).$$

This, of course, holds for every $h \in A_\sigma$. Let $\psi_\sigma(x)$ denote the set of all the ξ_σ fulfilling (12). We remark that if $\sigma, \tau \in \mathbb{N}^N$, then there exists $\rho \in \mathbb{N}^N$ such that $A_\sigma \cup A_\tau \subset A_\rho$; hence $\psi_\rho(x) \subset \psi_\sigma(x) \cap \psi_\tau(x)$. And as $\psi_\sigma(x)$ are weakly* closed and lie in B_{X^*} , we get that the intersection

$$\bigcap \{\psi_\sigma(x) : \sigma \in \mathbb{N}^N\}$$

is nonempty. Take some ξ in this set. Then

$$D^+f(x, h) \geq \langle \xi, h \rangle \geq D_+f(x, h) \quad \text{for all } h \in X.$$

since $\bigcup \{A_\sigma : \sigma \in \mathbb{N}^N\} = X$.

It remains to prove the claim. So fix $n, p \in \mathbb{N}$ and let E be a nonempty subset of B_{X^*} . Let $\tilde{E}$ denote the weakly* closed convex hull of E . Following the proof of [7, Theorem 12], let F be a weakly* closed convex subset of B_{Y^*} , minimal with respect to the inclusion, and such that

$$\tilde{E} = \{\xi/X : \xi \in F\},$$

where ξ/X means the restriction of ξ to the space X . As Z is Asplund, there is $z \in Z$ and $\beta > 0$ such that the slice $S(T^*(F), z, \beta)$ is nonempty and

$$\text{diam}_{B_Z} S(T^*(F), z, \beta) < 1/(np).$$

Then $S(F, Tz, \beta) \neq \emptyset$ and we estimate

$$\begin{aligned} \text{diam}_{nT(B_Z)} S(F, Tz, \beta) &= \sup\{\langle \xi_1 - \xi_2, v \rangle : \xi_i \in S(F, Tz, \beta), v \in nT(B_Z)\} \\ &= \sup\{\langle T^*\xi_1 - T^*\xi_2, u \rangle : T^*\xi_i \in S(T^*(F), z, \beta), u \in nB_Z\} \\ &= n \text{diam}_{B_Z} S(T^*(F), z, \beta) < n/(np) = 1/p. \end{aligned}$$

Now the set

$$E_1 = \{\xi/X : \xi \in F \setminus S(F, Tz, \beta)\}$$

is weakly* closed, convex, and, by the minimality, $E_1 \neq \tilde{E}$. So there are $0 \neq e \in X$ and $\alpha > 0$ such that $S(\tilde{E}, e, \alpha) \neq \emptyset$ and

$$S(\tilde{E}, e, \alpha) \cap E_1 = \emptyset.$$

Then $\xi \in S(F, Tz, \beta)$ whenever $\xi \in F$ and $\xi/X \in S(\tilde{E}, e, \alpha)$. Hence

$$\begin{aligned} \text{diam}_{A_{n,p}} S(E, e, \alpha) &\leq \text{diam}_{A_{n,p}} S(\tilde{E}, e, \alpha) \\ &= \sup\{\langle \eta_1 - \eta_2, v \rangle : \eta_i \in S(\tilde{E}, e, \alpha), v \in [nT(B_Z) + \tfrac{1}{p}B_Y] \cap X\} \\ &\leq \sup\{\langle \xi_1 - \xi_2, v \rangle : \xi_i \in S(F, Tz, \beta), v \in nT(B_Z) + \tfrac{1}{p}B_Y\} \\ &\leq \sup\{\langle \xi_1 - \xi_2, v \rangle : \xi_i \in S(F, Tz, \beta), v \in nT(B_Z)\} + 2/p \\ &= \text{diam}_{nT(B_Z)} S(F, Tz, \beta) + 2/p < 3/p. \end{aligned}$$

Finally we recall that $\tilde{E}$ is a weakly* closed convex hull of E , so $S(\tilde{E}, e, \alpha) \neq \emptyset$ implies that $S(E, e, \alpha)$ is also nonempty. This finishes the proof of the claim.

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