REMARKS ON PARTITIONER ALGEBRAS

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ABSTRACT. Partitioner algebras are defined in [1] and are a natural tool for studying the properties of maximal almost disjoint families of subsets of ω . We answer negatively two questions which were raised in [1]. We prove that there is a model in which the class of partitioner algebras is not closed under quotients and that it is consistent that there is a Boolean algebra of cardinality $\aleph_1$ which is not a partitioner algebra.

For any almost disjoint family $\mathfrak{M}$ of subsets of ω , a set $X \subset \omega$ is a *partitioner* of $\mathfrak{M}$ if $(\forall M \in \mathfrak{M}) (M \subset^* X \vee M \cap X =^* \emptyset)$. Recall that for sets A and B , $A \subset^* B$ if the set $A \setminus B$ is finite and $A =^* B$ if $A \subset^* B$ and $B \subset^* A$. The set of partitioners of $\mathfrak{M}$ forms a Boolean algebra $\mathfrak{P}_{\mathfrak{M}}$. Let $\mathcal{I}_{\mathfrak{M}}$ be the ideal in $\mathfrak{P}_{\mathfrak{M}}$ generated by $\mathfrak{M}$. The *partitioner algebra* $\mathfrak{B}_{\mathfrak{M}}$, corresponding to $\mathfrak{M}$ is then the quotient $\mathfrak{P}_{\mathfrak{M}}/\mathcal{I}_{\mathfrak{M}}$. An algebra $\mathfrak{B}$ is said to be *representable* if it is isomorphic to a partitioner algebra for some *mad* (maximal almost disjoint) family $\mathfrak{M}$. Similarly an algebra $\mathfrak{B}$ is said to be *weakly representable* if it is embeddable in a representable algebra. We will say that a set Y *splits* a set M if $M \cap Y$ and $M \setminus Y$ are both infinite. Note that $Y \notin \mathfrak{B}_{\mathfrak{M}}$ if and only if Y splits some member of $\mathfrak{M}$. See [1] for more details. If S is any countably infinite set, we generalize each of the above notions to apply to subsets of S in the obvious manner.

1. HOMOMORPHIC IMAGES

In this section we show that it is consistent with Martin's Axiom that the class of representable algebras is not closed under homomorphic images. It is shown in [1] that it follows from *CH* that every algebra of cardinality $\mathfrak{c}$ is representable.

Recall that $\mathfrak{b}$ is the minimum cardinality of an unbounded (mod finite) family of functions from ω to ω . We let $\mathcal{P}(X)$ denote the power set of X . For a set X , a family $\mathfrak{S} \subset \mathcal{P}(X)$ is *splitting* if it splits every infinite $Y \subset X$. Then $\mathfrak{s}$ is the minimum cardinality of a family of subsets of ω which

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is splitting. A family $\mathfrak{A} \subset \mathcal{P}(X)$ is *independent* if for each disjoint pair of finite subfamilies $\mathfrak{H}$ and $\mathfrak{J}$ of $\mathfrak{A}$, $\bigcap \mathfrak{H} \cap \bigcap_{J \in \mathfrak{J}} [X - J]$ is infinite.

Theorem 1. *If $\mathfrak{b} \leq \mathfrak{s}$, then the free algebra with $\mathfrak{b}$ generators is weakly representable.*

Proof. Fix an unbounded family $\{f_\alpha | \alpha < \mathfrak{b}\} \subset {}^\omega \omega$ of increasing functions and an independent family $\{A_\alpha | \alpha < \mathfrak{b}\} \subset \mathcal{P}(\omega)$. We assume also that for $\alpha < \beta < \mathfrak{b}$, $\{n | f_\alpha(n) \geq f_\beta(n)\}$ is finite. For each $\alpha < \mathfrak{b}$, define a subset of $\omega \times \omega$,

$$X_\alpha = [A_\alpha \times \omega] \setminus \{(n, m) | m < f_\alpha(n)\}.$$

Claim. There is a maximal almost disjoint family $\mathfrak{M}$ of infinite subsets of $\omega \times \omega$ such that $\{X_\alpha | \alpha < \mathfrak{b}\} \subset \mathfrak{P}_{\mathfrak{M}}$.

Proof of Claim. Let Y be an infinite subset of $\omega \times \omega$; we will show that there is an infinite subset of Y which is not split by the family $\{X_\alpha | \alpha < \mathfrak{b}\}$. Indeed, if there is an n such that $Y \cap (\{n\} \times \omega)$ is infinite, then this is trivial. On the other hand, if Y is finite in each “column,” we define $g \in {}^\omega \omega$ such that for each $n \in \omega$, $Y \cap (\{n\} \times \omega) \subset \{n\} \times g(n)$. Now choose $\beta < \mathfrak{b}$, such that $\{n | g(n) < f_\beta(n)\}$ is infinite (since $\{f_\beta | \beta < \mathfrak{b}\}$ is unbounded). Let $Y' = \{(n, m) \in Y | m < f_\beta(n)\}$. Since we are assuming $\mathfrak{b} \leq \mathfrak{s}$, the family $\{X_\alpha \cap Y' | \alpha < \beta\}$ is not a splitting family on $\mathcal{P}(Y')$. Therefore there is an infinite $M \subset Y' \subset Y$ which is not split by this family. This completes the proof of the claim, since $X_\gamma \cap Y'$ is finite for $\gamma > \beta$.

Therefore we may choose a maximal almost disjoint family $\mathfrak{M}$ of subsets of $\omega \times \omega$ so that no member of $\mathfrak{M}$ is split by any X_α . Therefore $\{X_\alpha | \alpha < \mathfrak{b}\} \subset \mathfrak{P}_{\mathfrak{M}}$. Since $\{X_\alpha | \alpha < \mathfrak{b}\}$ is an independent family, it follows that no infinite member of the algebra generated by it is in $\mathfrak{I}_{\mathfrak{M}}$ (although this would be easy to guarantee by refining $\mathfrak{M}$). Clearly, then $\mathfrak{B}_{\mathfrak{M}}$ contains the free algebra on $\mathfrak{b}$ generators.

We will need the following result.

Proposition 2 [3]. *It is consistent with $MA + \mathfrak{c} > \aleph_1$, that $\mathcal{P}(\omega_1)$ is not representable.*

We can now solve Problem 2 from [1].

Corollary 3. *It is consistent with MA , that there is a representable algebra with a homomorphic image which is not representable.*

Proof. Since $MA + \mathfrak{c} > \aleph_1$ implies that $2^{\aleph_1} = \mathfrak{c}$, it also implies that $\mathcal{P}(\omega_1)$ is a homomorphic image of the free algebra on $\mathfrak{c}$ generators. Let $\mathfrak{M}$ be chosen as in Theorem 1 so that $\mathfrak{B} = \mathfrak{B}_{\mathfrak{M}}$ contains the free algebra on $\mathfrak{c}$ generators as a subalgebra. Call this subalgebra $\mathfrak{C}$ and let φ be the homomorphism to $\mathcal{P}(\omega_1)$. For each $\xi < \omega_1$, fix an ultrafilter $\mathfrak{U}_\xi$ on $\mathfrak{B}$ so that $B \in (\mathfrak{U}_\xi \cap \mathfrak{C}) \Rightarrow \xi \in \varphi(B)$. Now define a function ψ from $\mathfrak{B}$ to $\mathcal{P}(\omega_1)$ by $\psi(B) = \{\xi \in \omega_1 | B \in \mathfrak{U}_\xi\}$.

We claim that ψ is a homomorphism from $\mathfrak{B}$ onto $\mathcal{P}(\omega_1)$. To see that it is onto, let $Y \in \mathcal{P}(\omega_1)$ and note that each $B \in \varphi^{-1}(Y)$ maps onto Y by ψ .

This completes the proof.

2. NONREPRESENTABLE ALGEBRAS OF SIZE $\aleph_1$

In this section we prove that if the splitting number, $\mathfrak{s}$, is less than $\mathfrak{b}$ then there is an algebra of size $\mathfrak{s}$ which cannot be represented. It has been shown to be consistent that $\mathfrak{s} = \aleph_1 < \mathfrak{b} = \aleph_2$ in [4]. We will also show that under these hypotheses, $\mathcal{P}(\omega)$ cannot be represented.

Theorem 4. *If $\mathfrak{s} < \mathfrak{b}$, the algebra $\mathfrak{B}$ (generated by a small splitting family together with the finite sets) is not even weakly representable.*

Proof. Assume that $\mathfrak{s} < \mathfrak{b}$. Let $\{A_\alpha \mid \alpha \in \mathfrak{s}\} \subset \mathcal{P}(\omega)$ be a splitting family. Let $\mathfrak{B}$ be the subalgebra of $\mathcal{P}(\omega)$ generated by the splitting family together with the finite sets. We shall show that $\mathfrak{B}$ cannot be represented.

Indeed, suppose that $\mathfrak{M}$ is a *mad* family which represents $\mathfrak{B}$. Let φ denote the embedding of $\mathfrak{B}$ into $\mathfrak{B}_{\mathfrak{M}}$.

Let $X_0 = \varphi(\{0\})$ and for $n \in \omega$, let

$$X_{n+1} = \varphi(\{n+1\}) \setminus \left[\bigcup_{k \leq n} X_k \right].$$

Since φ is an embedding, for each $b \in \mathfrak{B}$, there is a sequence $\{I(b, n) \mid n \in \omega\} \subset \mathfrak{I}_{\mathfrak{M}}$, such that

$$n \in b \Rightarrow X_n \setminus \varphi(b) \subset I(b, n)$$

and

$$n \notin b \Rightarrow X_n \cap \varphi(b) \subset I(b, n).$$

Since the minimum cardinality of a maximal almost disjoint family of subsets of ω , $\mathfrak{a}$, is at least $\mathfrak{b}$, we may choose $J_n \subset X_n$, such that $J_n \in \mathfrak{M}$ and $J_n \cap I(b, n)$ is finite for each $b \in \mathfrak{B}$.

Hence, for each $\alpha < \mathfrak{s}$, there is an $f_\alpha \in {}^\omega \omega$, so that

$$n \in A_\alpha \Rightarrow \varphi(A_\alpha) \supset J_n \setminus f_\alpha(n),$$

and

$$n \notin A_\alpha \Rightarrow \varphi(A_\alpha) \cap (J_n \setminus f_\alpha(n)) = \emptyset.$$

Since $\mathfrak{s} < \mathfrak{b}$, the family $\{f_\alpha : \alpha < \mathfrak{s}\}$ is bounded in ${}^\omega \omega$, so we may choose $h \in {}^\omega \omega$, so that $f_\alpha <^* h$ for each $\alpha < \mathfrak{s}$. Now we may choose $M \in \mathfrak{M} \setminus \{J_n : n \in \omega\}$ so that $M \cap \bigcup_{n \in \omega} J_n - h(n)$ is infinite. Let $A = \{n \mid J_n \cap M \neq \emptyset\}$ and choose $\alpha < \mathfrak{s}$ so that A_α splits A —that is, $|A \cap A_\alpha| = |A \setminus A_\alpha|$. We claim that $\varphi(A_\alpha)$ splits M , which contradicts the fact that $\varphi(A_\alpha)$ is a partitioner. Indeed, let $m \in \omega$ be such that $h(n) > f_\alpha(n)$ for each $n \geq m$. Now it is routine to verify that

$$\varphi(A_\alpha) \cap M \supset \bigcup_{n \in (A-m) \cap A_\alpha} [M \cap J_n],$$

and

$$M \setminus \varphi(A_\alpha) \supset \bigcup_{n \in (A-m) \setminus A_\alpha} [M \cap J_n];$$

which are both infinite.

We now answer Problem 4 from [1].

Corollary 5. $\mathfrak{s} = \aleph_1 < \mathfrak{b}$ implies there is an algebra of cardinality $\aleph_1$ which is not representable.

It turns out that this provides another model in which the solution to Problem 2 of [1] is negative. It is proven in [2] that the free algebra with $\aleph_1$ generators is representable. Therefore we obtain the following result as a corollary.

Corollary 6. If $\mathfrak{s} = \aleph_1 < \mathfrak{b}$, then the class of representable algebras is not closed under homomorphic images.

Let us also remark that the above proof also proves the following.

Corollary 7. If $\mathfrak{s} < \mathfrak{b}$, then $\mathcal{P}(\omega)$ is not weakly representable.

3. QUESTIONS

- (1) If $\kappa < \mathfrak{s}$, then the free algebra with κ generators is easily seen to be weakly representable—is it representable?
- (2) Is the free algebra on $\mathfrak{s}$ generators always weakly representable?
- (3) Does the representability of $\mathcal{P}(\omega)$ imply that $\mathfrak{s} = \mathfrak{c}$? (It does not imply that $\mathfrak{p} = \mathfrak{c}$.)
- (4) Does the representability of $\mathcal{P}(\omega)$ imply that $\mathcal{P}(\omega)/fin$ is representable?
- (5) Does the representability of $\mathcal{P}(\omega)$ imply that all algebras of size at most $\mathfrak{c}$ are representable?
- (6) Can Theorem 4 be proven even if the algebra $\mathfrak{B}$ is not made to be atomic?

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