HOMOGENEOUS POLYNOMIALS AND INVARIANT SUBSPACES IN THE POLYDISC. II

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ABSTRACT. We determine the invariant subspaces M of $L^2(T^2)$ for which there is a subspace S of M and a positive integer r such that

$$M = \sum_{n=0}^{\infty} \bigoplus \left[\sum_{j=0}^n z^j w^{r(n-j)} S \right],$$

where, for a subspace A of $L^2(T^2)$, $[A]$ denotes the closure of A .

1. INTRODUCTION

Let $L^2 = L^2(T^2)$ be the Lebesgue space on the torus T^2 , the product of two unit circles. A closed subspace M of L^2 is said to be invariant if $zM \subseteq M$ and $wM \subseteq M$, where z and w are the coordinate functions on T^2 . In a previous paper [3], the first author considered homogeneous decompositions for invariant subspaces (indeed, for invariant subspaces on a general torus) and determined invariant subspaces having such decompositions. In this paper we continue the study of [3].

Let M be an invariant subspace of L^2 and let r be a positive integer. For $n = 0, 1, 2, \dots$, put

$$M_n = \left[\sum_{j=0}^n z^j w^{r(n-j)} M \right],$$

where for a set A of L^2 , $[A]$ denotes the closed linear span in L^2 . Then M_n is invariant and

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq \cdots.$$

Thus, setting $S_n = M_n \ominus M_{n+1}$ for $n = 0, 1, 2, \dots$, we obtain the following

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decomposition:

$$(*) \quad M = \left(\sum_{n=0}^{\infty} \bigoplus S_n \right) \oplus M_{\infty},$$

where $M_{\infty} = \bigcap_{n=0}^{\infty} M_n$, which is an invariant subspace with $M_{\infty} = [zM_{\infty} + w^r M_{\infty}]$. Clearly, for $n = 0, 1, 2, \dots$,

$$zS_n \subseteq \left(\sum_{j=n+1}^{\infty} \bigoplus S_j \right) \oplus M_{\infty} \quad \text{and} \quad w^r S_n \subseteq \left(\sum_{j=n+1}^{\infty} \bigoplus S_j \right) \oplus M_{\infty}.$$

Definition. Let M be an invariant subspace of L^2 and r a positive integer. If

$$zS_n \subseteq S_{n+1} \quad \text{and} \quad w^r S_n \subseteq S_{n+1}$$

for $n = 0, 1, 2, \dots$, then the decomposition $(*)$ is said to be r -homogeneous. If in addition $M_{\infty} = \{0\}$, then we call M an r -homogeneous invariant subspace.

It is easy to see that $M = [zH^2 + w^r H^2]$ is an r -homogeneous invariant subspace, where $H^2 = H^2(T^2)$ is the space of all f in L^2 whose Fourier coefficients $\hat{f}(j, k)$ are zero for every pair (j, k) such that $j < 0$, or $k < 0$. We characterize r -homogeneous invariant subspaces. The case of $r = 1$ was considered in [3], but the general case requires a new idea.

2. RESULTS

Lemma 1. Let M be an invariant subspace of L^2 . If the decomposition $(*)$ is r -homogeneous, then the following relations hold.

- (1) $w^j S_0 \subseteq S_0 \oplus S_1$ for $j = 0, 1, 2, \dots, r-1$.
- (2) $S_n = [\sum_{j=0}^n z^j w^{r(n-j)} S_0]$ for $n = 0, 1, 2, \dots$.

Proof. Since the decomposition $(*)$ is r -homogeneous, $w^r S_0$ is orthogonal to

$$M_2 = [z^2 M + zw^r M + w^{2r} M].$$

Hence, for $j = 0, 1, 2, \dots, r-1$, $w^j S_0 \perp \overline{w^{r-j}} M_2$. Then, since $\overline{w^{r-j}} M_2 \supseteq M_2$, we have $w^j S_0 \perp M_2$ and $w^j S_0 \subseteq S_0 \oplus S_1$. This shows (1). The assumption that $(*)$ is r -homogeneous also yields $\sum_{j=0}^n z^j w^{r(n-j)} S_0 \subseteq S_n$ for $n = 0, 1, 2, \dots$. Since

$$\sum_{j=0}^{n+1} z^j w^{r(n+1-j)} M = \sum_{j=0}^n z^j w^{r(n-j)} (zM + w^r M),$$

it follows that

$$\left[\sum_{j=0}^n z^j w^{r(n-j)} S_0 \right] \oplus \left[\sum_{j=0}^{n+1} z^j w^{r(n+1-j)} M \right] = \left[\sum_{j=0}^n z^j w^{r(n-j)} M \right].$$

This shows (2).

For a positive integer r , let $\mathcal{L}_r$ denote the subspace of all functions f in L^2 whose Fourier coefficients $\hat{f}(j, k)$ are zero for every pair (j, k) such that $rj + k \neq 0$. Thus the subspace $\mathcal{L}_r$ is the closure in L^2 of polynomials in $\bar{z}w^r$ and $z\bar{w}^r$. Let S be a subspace of $\sum_{j=0}^{r-1} \oplus w^j \mathcal{L}_r$ and let H_S denote the invariant subspace generated by S . The space S is orthogonal to $[zS + w^r S]$. If

$$wS \subseteq S \oplus [zS + w^r S]$$

(note that this inclusion always holds for $r = 1$), then H_S has the decomposition

$$H_S = \sum_{n=0}^{\infty} \oplus \left[\sum_{j=0}^n z^j w^{r(n-j)} S \right].$$

Thus H_S is r -homogeneous and

$$S_0 := H_S \ominus [zH_S + w^r H_S] = S.$$

Theorem 1. If M is an r -homogeneous invariant subspace of L^2 then $M = qH_S$, where q is a unimodular function on T^2 and S is a subspace of $\sum_{j=0}^{r-1} \oplus w^j \mathcal{L}_r$ such that

$$wS \subseteq S \oplus [zS + w^r S].$$

Theorem 1 follows from Lemma 2 and Theorem 2 below.

Lemma 2. Let M be an r -homogeneous invariant subspace and

$$\widetilde{M} = \left[\bigcup_{k=0}^{\infty} (\bar{z}w^r)^k M \right].$$

Then $\widetilde{M}$ is an r -homogeneous invariant subspace. If

$$M = \sum_{n=0}^{\infty} \oplus S_n \quad \text{and} \quad \widetilde{M} = \sum_{n=0}^{\infty} \oplus \widetilde{S}_n$$

are the r -homogeneous decompositions of M and $\widetilde{M}$, respectively, then

$$\widetilde{S}_n = \left[\sum_{k=0}^{\infty} (\bar{z}w^r)^k S_n \right]$$

for $n = 0, 1, 2, \dots$.

Proof. Obviously $\widetilde{M}$ is invariant. Let $R_n = [\bigcup_{k=0}^{\infty} (\bar{z}w^r)^k S_n]$ for $n = 0, 1, 2, \dots$. Since M is r -homogeneous, if $m \neq n$, then $(\bar{z}w^r)^k S_m \perp (\bar{z}w^r)^j S_n$ for $k, j = 0, 1, 2, \dots$, so $R_m \perp R_n$. Hence we have $\widetilde{M} = \sum_{n=0}^{\infty} \oplus R_n$. Since $S_n = [\sum_{j=0}^n z^j w^{r(n-j)} S_0]$ by Lemma 1, we also have $R_n = [\sum_{j=0}^n z^j w^{r(n-j)} R_0]$. Thus $\widetilde{S}_n = R_n$ for $n = 0, 1, 2, \dots$. It is easy to see that $zR_n \subseteq R_{n+1}$ and $w^r R_n \subseteq R_{n+1}$ for $n = 0, 1, 2, \dots$. Therefore $\widetilde{M}$ is r -homogeneous.

For a positive integer r , let $\mathcal{H}_r$ denote the closure in L^2 of the polynomials in $\overline{z}w^r$, a subspace of $\mathcal{L}_r$.

Theorem 2. Let M be an r -homogeneous invariant subspace of L^2 such that $\overline{z}w^r M \subseteq M$.

(1) If $\overline{z}w^r M = M$, then $M = qH_S$ where q is a unimodular function on T^2 and $S = \chi_E(\sum_{j=0}^{r-1} \oplus w^j \mathcal{L}_r)$ for some Borel set E of T^2 whose characteristic function χ_E belongs to $\mathcal{L}_r$.

(2) If $\overline{z}w^r M \subsetneq M$ then $M = qH_S$, where q is a unimodular function on T^2 and

$$S = \left(\sum_{0 \leq j \leq i-1} \oplus w^j \mathcal{H}_r \right) \oplus \left(\sum_{1 \leq k \leq r-i} \oplus \overline{z}w^k \mathcal{H}_r \right)$$

for some integer i with $1 \leq i \leq r$.

If an invariant subspace M satisfies $\overline{z}w^r M \subseteq M$, then for $n = 1, 2, \dots$,

$$\left[\sum_{j=0}^n z^j w^{r(n-j)} M \right] = z^n M$$

and hence the decomposition (*) for M coincides with the Wold decomposition of M with respect to the isometry of multiplication by z on M , i.e.,

$$S_n = z^n(M \ominus zM)$$

for $n = 0, 1, 2, \dots$. If, moreover, the decomposition (*) is r -homogeneous, then $w^r(M \ominus zM) \subseteq z(M \ominus zM)$. Conversely, we see that the condition $w^r(M \ominus zM) \subseteq z(M \ominus zM)$ for an invariant subspace M implies that $\overline{z}w^r M \subseteq M$ and M has an r -homogeneous decomposition. Thus Theorem 2 should be compared with the results of [1, 2], in which the condition $w(M \ominus zM) \subseteq M \ominus zM$ for an invariant subspace M was considered to characterize an invariant subspace of the form qH^2 .

In the following we prove Theorem 2. Let M be an r -homogeneous invariant subspace such that $\overline{z}w^r M \subseteq M$, and let $M = \sum_{n=0}^{\infty} \oplus S_n$ be the r -homogeneous decomposition of M . Then, as seen above, $S_n = z^n S_0$ for $n = 1, 2, \dots$ and $\overline{z}w^r S_0 \subseteq S_0$. Since M is equal to the invariant subspace generated by S_0 by Lemma 1, in order to prove Theorem 2, it suffices to show that the subspace $\overline{q}S_0$, for some unimodular function q , is equal to one of the subspaces S in Theorem 2.

Let $S_0^{(1)} = \{f \in S_0 : w^{r-1}f \in S_0\}$. (Note that if $r = 1$, then $S_0^{(1)} = S_0$.)

Since Theorem 2 was proved in [3] in the case of $r = 1$, let $r \geq 2$ in the following.

Lemma 3. (1) $S_0^{(1)} \neq \{0\}$.

(2) $\overline{z}w^r S_0^{(1)} \subseteq S_0^{(1)}$.

(3) $w^j S_0^{(1)} \subseteq S_0$ for $1 \leq j \leq r-1$.

(4) $w^i S_0^{(1)} \perp w^j S_0^{(1)}$ for $0 \leq i < j \leq r-1$.

Proof. (1) Since $w^r S_0 \subseteq S_1$, it follows that $w S_0 \not\subseteq S_0$. Let f be a function in S_0 such that $wf \notin S_0$. As remarked above, $S_1 = zS_0$. It follows from (1) of Lemma 1 that

$$wf = g + zh$$

for some functions g and h in S_0 with $h \neq 0$, so

$$w^{r-1}g + zw^{r-1}h = w^r f \in zS_0.$$

Then, since $w^{r-1}g$ and $w^{r-1}h$ belong to $S_0 \oplus zS_0$ by Lemma 1, $zw^{r-1}h \in zS_0$ and so $w^{r-1}h \in S_0$. Hence h is a nonzero element of $S_0^{(1)}$.

(2) follows from the inclusion $\bar{z}w^r S_0 \subseteq S_0$.

(3) For $1 \leq j \leq r-1$, since $zw^{r-1-j}M \subseteq zM$, $w^{r-1}S_0^{(1)}$ and $zw^{r-1-j}M$ are orthogonal, hence $w^j S_0^{(1)} \perp zM$. This shows that $w^j S_0^{(1)} \subseteq S_0$.

(4) For $i < j$, since $w^r S_0^{(1)} \subseteq zS_0$ and $w^{r-j+i} S_0^{(1)} \subseteq S_0$ by (3),

$$w^{r-j+i} S_0^{(1)} \perp w^r S_0^{(1)},$$

and so $w^i S_0^{(1)} \perp w^j S_0^{(1)}$.

Lemma 4. (1) If $\bar{z}w^r S_0^{(1)} = S_0^{(1)}$, then $S_0^{(1)} = q\chi_E \mathcal{L}_r$ where q is unimodular and $\chi_E \in \mathcal{L}_r$.

(2) If $\bar{z}w^r S_0^{(1)} \subsetneq S_0^{(1)}$, then $S_0^{(1)} = q\mathcal{H}_r$ where q is unimodular.

Using Lemma 3, the proof is completed by the same argument as in the proofs of [3, Theorem 3 and Proposition 5].

By Lemma 3, we have $S_0 \supseteq \sum_{j=0}^{r-1} \oplus w^j S_0^{(1)}$.

Lemma 5. Let $S_0^{(2)} = S_0 \ominus (\sum_{j=0}^{r-1} \oplus w^j S_0^{(1)})$. If $f \in S_0^{(2)}$, then

$$f = \sum_{j=1}^r z\bar{w}^j g_j,$$

where $g_j \in S_0^{(1)} \ominus \bar{z}w^r S_0^{(1)}$ for $1 \leq j \leq r$ and the functions g_j ($1 \leq j \leq r$) satisfy

$$w^k \left(\sum_{j=k+1}^r z\bar{w}^j g_j \right) \in S_0^{(2)}$$

for $0 \leq k \leq r-1$.

Proof. By Lemma 1, we have $wf = f_1 + zg_1$ for some f_1 and g_1 in S_0 . Then we see that $f_1 \in S_0^{(2)}$ and $g_1 \in S_0^{(1)} \ominus \bar{z}w^r S_0^{(1)}$. In fact, since $f \in S_0^{(2)}$, wf is orthogonal to $w^j S_0^{(1)}$ for $1 \leq j \leq r$. Hence $f_1 \perp w^j S_0^{(1)}$ for $1 \leq j \leq r-1$ and $g_1 \perp \bar{z}w^r S_0^{(1)}$. Since $w^r f \in zS_0$ and $w^{r-1} S_0^{(1)} \subseteq S_0$, it follows that $wf \perp S_0^{(1)}$ and so $f_1 \perp S_0^{(1)}$. Thus $f_1 \in S_0^{(2)}$. The proof of (1) in Lemma 3 shows that $g_1 \in S_0^{(1)}$, hence $g_1 \in S_0^{(1)} \ominus \bar{z}w^r S_0^{(1)}$.

Now, assume that for some $k \geq 1$,

$$(**) \quad w^k f = f_k + \sum_{j=1}^k zw^{k-j} g_j,$$

where $g_j \in S_0^{(1)} \ominus \bar{z}w^r S_0^{(1)}$ for $1 \leq j \leq k$ and $f_k \in S_0^{(2)}$. Then, using the argument given above, we have $wf_k = f_{k+1} + zg_{k+1}$ for $f_{k+1} \in S_0^{(2)}$ and $g_{k+1} \in S_0^{(1)} \ominus \bar{z}w^r S_0^{(1)}$. Consequently,

$$w^{k+1} f = f_{k+1} + \sum_{j=1}^{k+1} zw^{k+1-j} g_j.$$

Thus by induction, (**) holds for all $k \geq 1$. Since $w^r f \in zS_0$, it follows from (**) for $k = r$, that

$$f = \sum_{j=1}^r z\bar{w}^j g_j.$$

For $0 \leq k \leq r-1$, comparing (**) with

$$w^k f = w^k \left(\sum_{j=1}^k z\bar{w}^j g_j \right) + w^k \left(\sum_{j=k+1}^r z\bar{w}^j g_j \right),$$

we have

$$w^k \left(\sum_{j=k+1}^r z\bar{w}^j g_j \right) = f_k \in S_0^{(2)}.$$

This completes the proof.

Proof of Theorem 2. (1) By Lemma 5, we have $\bar{z}w^r S_0 \subseteq \sum_{j=0}^{r-1} \oplus w^j S_0^{(1)}$. Since $\bar{z}w^r S_0 = S_0$ by hypothesis, it follows that

$$S_0 = \sum_{j=0}^{r-1} \oplus w^j S_0^{(1)} \quad \text{and} \quad \bar{z}w^r S_0^{(1)} = S_0^{(1)}.$$

Then by Lemma 4, $S_0^{(1)} = q\chi_E \mathcal{L}_r$ where q is unimodular and $\chi_E \in \mathcal{L}_r$. Therefore we have

$$S_0 = q\chi_E \left(\sum_{j=0}^{r-1} \oplus w^j \mathcal{L}_r \right)$$

and $M = qH_S$ where $S = \chi_E (\sum_{j=0}^{r-1} \oplus w^j \mathcal{L}_r)$.

(2) If $\bar{z}w^r S_0^{(1)} = S_0^{(1)}$, then by Lemma 5 we have

$$S_0 = \sum_{j=0}^{r-1} \oplus w^j S_0^{(1)},$$

and hence $\bar{z}w^r S_0 = S_0$. Thus under our assumption, $\bar{z}w^r S_0^{(1)} \subsetneq S_0^{(1)}$. It follows from (2) of Lemma 4 that $S_0^{(1)} = q\mathcal{H}_r$ for some unimodular function q , and that $S_0^{(1)} \ominus \bar{z}w^r S_0^{(1)} = [q]$. Therefore, if

$$f \in S_0^{(2)} = S_0 \ominus \left(\sum_{j=0}^{r-1} \oplus w^j S_0^{(1)} \right),$$

then by Lemma 5 there are scalars α_j ($1 \leq j \leq r$) such that

$$f = q \left(\sum_{j=1}^r \alpha_j z \bar{w}^j \right)$$

and

$$w^k \left(q \sum_{j=k+1}^r \alpha_j z \bar{w}^j \right) \in S_0^{(2)}$$

for $0 \leq k \leq r-1$. Let s be the largest number among numbers j with $\alpha_j \neq 0$. Then the conditions $w^k (q \sum_{j=k+1}^s \alpha_j z \bar{w}^j) \in S_0^{(2)}$, for $0 \leq k \leq s-1$, imply $qz\bar{w}^j \in S_0^{(2)}$ for $1 \leq j \leq s$ and so $s \leq r-1$. Thus we have

$$S_0^{(2)} = q[\{z\bar{w}^j : j = 1, 2, \dots, s\}]$$

for some $s \leq r-1$ or $S_0^{(2)} = \{0\}$. It follows that

$$\bar{q}S_0 = \left(\sum_{0 \leq j \leq r-1-s} \oplus w^j \mathcal{H}_r \right) \oplus \left(\sum_{1 \leq k \leq s} \oplus z \bar{w}^k \mathcal{H}_r \right)$$

for some $s = 0, 1, 2, \dots, r-1$, and that $M = qH_S$ where $S = \bar{q}S_0$.

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