ON THE SENDOV CONJECTURE FOR SIXTH DEGREE POLYNOMIALS

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ABSTRACT. The Sendov conjecture asserts that if $p(z) = \prod_{k=1}^n (z - z_k)$ is a polynomial with zeros $|z_k| \leq 1$, then each disk $|z - z_k| \leq 1$, $(1 \leq k \leq n)$ contains a zero of $p'(z)$. This conjecture has been verified in general only for polynomials of degree $n = 2, 3, 4, 5$. If $p(z)$ is an extremal polynomial for this conjecture when $n = 6$, it is known that if a zero $|z_j| \leq \lambda_6 = 0.626997 \dots$ then $|z - z_j| \leq 1$ contains a zero of $p'(z)$. (The conjecture for $n = 6$ would be proved if $\lambda_6 = 1$.) It is shown that λ_6 can be improved to $\lambda_6 = 63/64 = 0.984375$.

The well-known conjecture of Sendov [6, Problem 4.5] posed in 1962 states that if $p(z) = \prod_{k=1}^n (z - z_k)$, $|z_k| \leq 1$ ($1 \leq k \leq n$), then each disk $|z - z_k| \leq 1$ ($1 \leq k \leq n$) contains a zero of $p'(z)$. Surprisingly this conjecture has been proved only for polynomials of degree $n \leq 5$ and in a few special cases [2-4, 9-15]. (See Marden [8] for an excellent expository article on this conjecture.) The case $n = 5$ was proved in 1969 [9], while the general case for $n = 6$ is still open.

The Sendov conjecture can be viewed as an extremal problem over a compact family of functions as follows. Let $\mathcal{P}_n$ denote the family of all monic polynomials of the form

$$p(z) = \prod_{k=1}^n (z - z_k), \quad |z_k| \leq 1 \quad (1 \leq k \leq n).$$

Thus, by the classical Gauss-Lucas Theorem we have

$$p'(z) = n \prod_{j=1}^{n-1} (z - \zeta_j), \quad |\zeta_j| \leq 1 \quad (1 \leq j \leq n-1).$$

Define $I(z_k) = \min_{1 \leq j \leq n-1} |z_k - \zeta_j|$, $I(p) = \max_{1 \leq k \leq n} I(z_k)$, and $I(\mathcal{P}_n) = \sup_{p \in \mathcal{P}_n} I(p)$. Phelps and Rodriguez [10] proved that there exists an extremal polynomial $p_*(z) = \prod_{k=1}^n (z - z_k^*) \in \mathcal{P}_n$ such that $I(p_*) = I(\mathcal{P}_n)$ (see Lemma

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A). It thus suffices to prove the conjecture for p_* i.e., show $I(z_k^*) \leq 1$ for $1 \leq k \leq n$. They also proved a result which implies that if $n \geq 5$ and z_j^* , for some j , satisfies $|z_j^*| \leq \lambda_n$, where λ_n is the unique root of $(1+x^2)(1+x)^{n-3} - n = 0$, then $I(z_j^*) \leq 1$. In the special case $n = 6$ their result gives $\lambda_6 = 0.626997 \dots$. Of course the conjecture for $n = 6$ would be proved if $\lambda_6 = 1$. The purpose of this paper is to improve the bound λ_6 :

Theorem. *If $p_*(z) = \prod_{k=1}^6 (z - z_k^*)$ is an extremal polynomial for $I(\mathcal{P}_6)$ and $|z_j^*| \leq 63/64$, for some $j = 1, 2, \dots, 6$, then the disk $|z - z_j^*| \leq 1$ contains a zero of $p'_*(z)$.*

This result now reduces the conjecture for the case $n = 6$ to considering the zeros of p_* close to $|z| = 1$. In this direction, we point out that Goodman, Rahman, and Ratti [5] have proved that if z_0 is a zero of $p(z)$ and $|z_0| = 1$, then the disk $|z - z_0/2| \leq \frac{1}{2}$ contains a zero of $p'(z)$. Thus for boundary zeros a result stronger than the Sendov conjecture holds.

Our proof of the main result involves a blend of analytic and geometric ideas. The method presented here can be used to increase the bounds for λ_n for all $n \geq 6$.

We make use of the following known results:

Lemma A (Phelps and Rodriguez [10]). *There exists an extremal polynomial $p_* \in \mathcal{P}_n$ such that $I(p_*) = I(\mathcal{P}_n)$. Moreover, $p_*(z)$ has a zero on each subarc of $|z| = 1$ of length π .*

Lemma B (Bojanov, Rahman, and Szynal [2]). *If $Q(z)$ is a monic polynomial of degree n , $Q(0) = 0$, and $Q'(z) \neq 0$ in $|z| \leq R$, then*

$$|Q(z)| > R^n - (R - \lambda)^n \quad \text{for } |z| = \lambda \leq R \sin \pi/n.$$

We also need the following estimates:

Lemma 1. *Let $p(z) = (z - a) \prod_{k=1}^{n-1} (z - z_k) \in \mathcal{P}_n$, $0 < a < 1$ and $p'(z) = n \prod_{j=1}^{n-1} (z - \zeta_j)$. If $I(a) > 1$, then*

$$\left| \frac{\zeta_j - a}{a\zeta_j - 1} \right| > \frac{1}{1 + a - a^2} \quad j = 1, 2, \dots, n-1.$$

Lemma 2. *If $p(z) = (z - a) \prod_{k=1}^{n-1} (z - z_k)$ is an extremal polynomial for $I(\mathcal{P}_n)$, $0 < a < 1$ and $p'(z) = n \prod_{j=1}^{n-1} (z - \zeta_j)$, then*

$$\prod_{j=1}^{n-1} \left| \frac{\zeta_j - a}{a\zeta_j - 1} \right| \leq \left[n - (n-3)a - \frac{4a^2}{1+a^2} \right]^{-1}.$$

Lemma 3. *Let $p(z) = (z - a) \prod_{k=1}^5 (z - z_k)$ be an extremal polynomial for $I(\mathcal{P}_6)$.*

(i) *If $0 \leq a < \frac{2}{3}$, then $I(a) \leq 1$.*

- (ii) If $\frac{2}{3} \leq a \leq \frac{63}{64}$ and $I(a) > 1$, then there exists a zero $\zeta_0 = a + \rho_0 e^{i\theta_0}$ of $p'(z)$ such that $\rho_0 > 1$ and

$$\cos \theta_0 \geq 1/25 - a.$$

Proof of Theorem. Suppose $p_*(z) = \prod_{k=1}^6 (z - z_k^*)$ is an extremal polynomial for $I(\mathcal{P}_6)$ and suppose that $|z_j^*| \leq 63/64$ for some j . By a rotation, if necessary, we may suppose that $z_j^* = a$, $0 \leq a \leq 63/64$. Hence

$$p_*(z) = (z - a) \prod_{k=1}^5 (z - z_k), \quad |z_k| \leq 1 \quad k = 1, 2, \dots, 5.$$

Assume by way of contradiction that $I(a) > 1$. By Lemma 3(i) we must then have $\frac{2}{3} \leq a \leq \frac{63}{64}$. Thus, making use of Lemma 3(ii) we can assert that there exists a critical point $\zeta_0 = a + \rho_0 e^{i\theta_0}$ with $\rho_0 > 1$ and $\cos \theta_0 \geq 1/25 - a$. It follows that

$$(1) \quad \left| \frac{\zeta_0 - a}{a\zeta_0 - 1} \right| \geq \frac{1}{[(1 - a^2)^2 + a^2 - 2a(1 - a^2) \cos \theta_0]^{1/2}} \\ \geq \frac{1}{[(1 - a^2)^2 + a^2 - 2a(1 - a^2)(1/25 - a)]^{1/2}}.$$

Assuming as we may that $|(\zeta_1 - a)/(a\zeta_1 - 1)| \leq |(\zeta_j - a)/(a\zeta_j - 1)|$ for the critical points $\zeta_j \neq \zeta_0$, we apply Lemma 2 with $n = 6$ to get

$$(2) \quad \left| \frac{\zeta_1 - a}{a\zeta_1 - 1} \right|^4 \left| \frac{\zeta_0 - a}{a\zeta_0 - 1} \right| \leq \prod_{j=1}^5 \left| \frac{\zeta_j - a}{a\zeta_j - 1} \right| \leq \frac{1}{6 - 3a - 4a^2/(1 + a^2)}.$$

Using (1) and (2) we obtain

$$(3) \quad \left| \frac{\zeta_1 - a}{a\zeta_1 - 1} \right|^4 \leq \frac{[(1 - a^2)^2 + a^2 - 2a(1 - a^2)(1/25 - a)]^{1/2}}{6 - 3a - 4a^2/(1 + a^2)} \equiv \phi(a).$$

Simple numerical calculations show that

$$\phi(a) \leq \frac{1}{(1 + a - a^2)^4} \quad \text{for } \frac{2}{3} \leq a \leq \frac{63}{64}.$$

From (3) we conclude that $|(\zeta_1 - a)/(a\zeta_1 - 1)| \leq 1/(1 + a - a^2)$, contradicting Lemma 1. $\square$

We now turn to the proofs of the lemmas.

Proof of Lemma 1. Assume by way of contradiction that $|(\zeta_1 - a)/(a\zeta_1 - 1)| \leq 1/(1 + a - a^2)$. Hence we get $(\zeta_1 - a)/(a\zeta_1 - 1) = re^{i\theta}$, where $0 \leq r \leq 1/(1 + a - a^2)$. This gives $\zeta_1 = (a - re^{i\theta})/(1 - are^{i\theta})$ and so

$$|\zeta_1 - a| = \frac{r(1 - a^2)}{|1 - are^{i\theta}|} \leq \frac{r(1 - a^2)}{1 - ar} \leq 1,$$

which contradicts $I(a) > 1$. $\square$

Proof of Lemma 2. Suppose that $p(z)$ is extremal and

$$(4) \quad p(z) = (z - a) \prod_{k=1}^{n-1} (z - z_k), \quad |z_k| \leq 1 \quad (1 \leq k \leq n-1),$$

$$(5) \quad p'(z) = n \prod_{j=1}^{n-1} (z - \zeta_j), \quad |\zeta_j| \leq 1 \quad (1 \leq j \leq n-1),$$

and $0 < a < 1$. Let $z = T(w) = (w - a)/(aw - 1)$ and note that

$$p(T(w)) = p_0(w)(aw - 1)^{-n},$$

where

$$p_0(w) = Aw(w^{n-1} + b_{n-1}w^{n-2} + \cdots + b_1).$$

The zeros of $p_0(w)$ are $0, w_1, w_2, \dots, w_{n-1}$, where $w_k = T(z_k)$, $1 \leq k \leq n-1$. Hence we see that

$$(6) \quad b_1 = (-1)^{n-1} \prod_{k=1}^{n-1} w_k$$

and

$$(7) \quad b_{n-1} = - \sum_{k=1}^{n-1} w_k.$$

Differentiating $p(T(w))$ gives

$$\frac{dp(T(w))}{dw} = \frac{dp(T(w))}{dz} \cdot \frac{dz}{dw} = A_{1/a} p_0(w) [-a(aw - 1)^{-n-1}],$$

where

$$(8) \quad A_{1/a} p_0(w) = np_0(w) + \left(\frac{1}{a} - w \right) p'_0(w) = B \prod_{j=1}^{n-1} (w - \gamma_j)$$

is the polar derivative of $p_0(w)$ with respect to $1/a$ (see Marden [7, p. 44]). Thus we get

$$(9) \quad p'(T(w)) = A_{1/a} p_0(w) [-a(aw - 1)^{-n-1} dw/dz]$$

(where $'$ denotes differentiation with respect to z). It follows from (5), (8), and (9) that the zeros of $p'(z)$ and $A_{1/a} p_0(w)$ are related by

$$(10) \quad \gamma_j = \frac{\zeta_j - a}{a\zeta_j - 1}, \quad 1 \leq j \leq n-1.$$

Next, we see after a simple check that

$$A_{1/a} p_0(w) = B \left[w^{n-1} + \cdots + \left(\frac{b_1}{n + ab_{n-1}} \right) \right] = B \prod_{j=1}^{n-1} (w - \gamma_j)$$

and so

$$(11) \quad \prod_{j=1}^{n-1} |\gamma_j| = \left| \frac{b_1}{n + ab_{n-1}} \right|.$$

By Lemma A there is a zero on each subarc of $|z| = 1$ of length π . Hence without loss of generality

$$\begin{cases} z_{n-1} = e^{i\theta_0}, & 0 \leq \theta_0 \leq \pi/2 \\ z_{n-2} = e^{i\theta_1}, & \theta_0 + \pi \leq \theta_1 \leq 2\pi. \end{cases}$$

(If $\operatorname{Im} z_{n-1} < 0$, simply consider $\overline{p(\bar{z})}$.) It is easy to check that

$$\begin{aligned} \operatorname{Re}\{T(z_{n-1}) + T(z_{n-2})\} &= \operatorname{Re}\left\{\left(\frac{e^{i\theta_0} - a}{ae^{i\theta_0} - 1}\right) + \left(\frac{e^{i\theta_1} - a}{ae^{i\theta_1} - 1}\right)\right\} \\ &\leq \frac{2a - (1 - a^2)\cos\theta_0}{(1 + a^2) - 2a\cos\theta_0} + \frac{2a + (1 + a^2)\cos\theta_0}{(1 + a^2) + 2a\cos\theta_0} \\ &= \frac{4a(1 + a^2)(1 - \cos^2\theta_0)}{(1 + a^2)^2 - 4a^2\cos^2\theta_0} \leq \frac{4a}{1 + a^2}. \end{aligned}$$

Using (6), (7), (11), and the above inequality, we obtain

$$\begin{aligned} \prod_{j=1}^{n-1} \left| \frac{\zeta_j - a}{a\zeta_j - 1} \right| &\leq \frac{1}{|n - a \sum_{k=1}^{n-1} T(z_k)|} \\ &\leq \frac{1}{n - a \sum_{k=1}^{n-1} \operatorname{Re} T(z_k)} \\ &\leq \frac{1}{n - 4a^2/(1 + a^2) - a(n - 3)}. \end{aligned}$$

The proof of the lemma is complete. $\square$

Note that (11) immediately gives $\prod_{j=1}^{n-1} |\gamma_j| \leq 1/(n - (n - 1)a)$. However this estimate is not good enough for our purposes.

Proof of Lemma 3. Let $p(z) = (z - a) \prod_{k=1}^5 (z - z_k)$ be extremal.

(i) $0 \leq a < \frac{2}{3}$. Clearly $I(0) \leq 1$ by the classical Gauss-Lucas Theorem. Next, from Lemma 2 with $n = 6$ we see that

$$(12) \quad \prod_{j=1}^5 \left| \frac{\zeta_j - a}{a\zeta_j - 1} \right| \leq \frac{1}{6 - 3a - 4a^2/(1 + a^2)}.$$

Assuming $|(\zeta_1 - a)/(a\zeta_1 - 1)| \leq |(\zeta_j - a)/(a\zeta_j - 1)|$ for $j = 1, 2, \dots, 5$ we have from (12) and an easy calculation that

$$\left| \frac{\zeta_1 - a}{a\zeta_1 - 1} \right| \leq \frac{1}{(6 - 3a - 4a^2/(1 + a^2))^{1/5}} \leq \frac{1}{1 + a - a^2}.$$

Thus, by Lemma 1, we get $I(a) \leq 1$.

(ii) $\frac{2}{3} \leq a \leq \frac{63}{64}$ and $I(a) > 1$. For fixed a we apply Lemma B with $Q(z) = p(z+a)$, $n = 6$, $R = 1$, and $\lambda = 1 - (1-a)^{1/6}$ to conclude that

$$(13) \quad |p(z)| > 1 - (1-\lambda)^6 = a \geq |p(0)|$$

for $|z-a| = \lambda = 1 - (1-a)^{1/6}$. Note that $\lambda \leq \frac{1}{2}$. Since $p'(z) \neq 0$ in $|z-a| \leq 1$ and the degree of $p(z)$ is six, we see that by Alexander's Theorem (See Marden [7, p. 110, exercise 2]) that $p(z)$ is univalent in $|z-a| \leq \frac{1}{2}$. Thus from (13) we see that there exists a *unique* z_0 such that $|z_0-a| \leq \lambda$ with $p(z_0) = p(0)$. We may assume that $\text{Im } z_0 \geq 0$ (if not, consider $\overline{p(\overline{z})}$). Let Γ_0 be the perpendicular bisector of the segment from 0 to z_0 , and let H_0^+ and H_0^- be the closed halfplanes bounded by Γ_0 . By a variant of the Grace-Heawood Theorem (see [1, Lemma 1] for example) we see that $p'(z)$ has a zero in both H_0^+ and H_0^- . Let $\omega_0 = \mu_0(x_0, y_0) + i\nu_0(x_0, y_0)$ be the intersection of Γ_0 with the circle $|z-a| = 1$ with $\nu_0 > 0$. It follows that $p'(z)$ has a zero $\zeta_0 = a + \rho_0 e^{i\theta_0}$ with $\rho_0 > 1$ and $\cos \theta_0 \geq \mu_0(x_0, y_0) - a$. (See Figure 1.) It suffices to prove that $\mu_0(x_0, y_0) \geq 1/25$.

FIGURE 1

Let $z^* = x^* + iy^*$ be the point on $|z - a| = \lambda$ such that the line through 0 and z^* is tangent to $|z - a| = \lambda$ and $y^* > 0$. Hence $x^* = (a^2 - \lambda^2)/a$. If $z_0 = r_0 e^{it_0}$ then by letting $z = re^{it_0} = x + iy$ ($r \leq r_0$) be the point on $|z - a| = \lambda$ with $y > 0$ and $x \leq x^*$, we conclude that $\mu_0(x_0, y_0) \geq \mu_0(x, y)$. Thus, it suffices to prove that $\mu_0(x, y) \geq 1/25$ where $(x - a)^2 + y^2 = \lambda^2$, $a - \lambda \leq x \leq x^*$, and $y \geq 0$.

A calculation shows that

$$(14) \quad \mu_0(x, y) = \frac{x(\lambda^2 + 3a^2) + 2a\beta - \sqrt{y^2(8ax + 4\beta - \beta^2)}}{2(2ax + \beta)},$$

where $\beta = \lambda^2 - a^2$. Observe that since $a - \lambda \leq x \leq x^*$ we get

$$y^2 = \lambda^2 - (x - a)^2 \leq 2\lambda(x - a + \lambda) \quad \text{and} \quad 2ax + \beta \leq -\beta.$$

From (14) we conclude that

$$\mu_0(x, y) \geq \frac{x(\lambda^2 + 3a^2) + 2a\beta - \sqrt{2\lambda(x - a + \lambda)(8ax + 4\beta - \beta^2)}}{-2\beta} \equiv \mu_1.$$

It is enough to show that $\mu_1 \geq 1/25$.

Now $\mu_1 \geq 1/25$ if and only if

$$(15) \quad F(x) \equiv c_1 x^2 + c_2 x + c_3 \geq 0, \quad a - \lambda \leq x \leq x^*,$$

where

$$\begin{aligned} c_1 &= (\lambda^2 + 3a^2)^2 - 16a\lambda, \\ c_2 &= 2\beta(2a + 0.08)(\lambda^2 + 3a^2) - 2\lambda\beta(4 - \beta) - 16a\lambda(\lambda - a), \\ c_3 &= \beta^2(2a + 0.08)^2 - 2\lambda\beta(4 - \beta)(\lambda - a), \end{aligned}$$

($\beta = \lambda^2 - a^2$). An easy check shows that $c_1 > 0$ and so (15) follows if $\Delta \equiv c_2^2 - 4c_1c_3 < 0$. A brief calculation shows that $\Delta = 4\lambda(\lambda - a)^2\Delta_0$, where

$$\begin{aligned} \Delta_0 &= [(\lambda + a)(4 - \beta) + 8a][8a\lambda + (\lambda + a)\{\lambda(4 - \beta) - (4a + 0.16)(\lambda^2 + 3a^2)\}] \\ &\quad + (\lambda + a)[16a(\lambda + a)(2a + 0.08)^2 + 2(4 - \beta)(\lambda^2 + 3a^2)^2 - 32a\lambda(4 - \beta)]. \end{aligned}$$

Finally, a computation shows that $\Delta_0 < 0$ for $\frac{2}{3} \leq a \leq \frac{63}{64}$. (It can easily be checked numerically that $\Delta_0 < -0.009$.) Thus (15) holds and hence $\mu_0 \geq 1/25$. $\square$

The proof of the Sendov conjecture has been elusive for more than twenty-five years and only verified in a few special cases. The method of proof in the case $n = 5$ does not seem to be useful for $n \geq 6$. It is not surprising to see the different ideas used to prove our results.

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