A GLOBAL PINCHING THEOREM FOR COMPACT MINIMAL SURFACES IN S^3

YI-JUNG HSU

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ABSTRACT. Let M be a compact minimally immersed surface in the unit sphere S^3 , and let S denote the square of the length of the second fundamental form of M . We prove that if $\|S\|_2 \leq 2\sqrt{2}\pi$, then M is either the equatorial sphere or the Clifford torus.

Let M be a compact minimally immersed hypersurface in the unit sphere S^{n+1} . Denote by S the square of the length of the second fundamental form of M . It is well known that if $0 \leq S \leq n$, then M is either the equatorial sphere or a Clifford torus [1]. Recently, C. L. Shen [4, Theorem 2] proved that if M is a compact embedded minimal surface of nonnegative Gauss curvature in the unit sphere S^3 with $\|S\|_2 < 1/(6912\sqrt{2\pi(g+1)})$, then M is the equatorial sphere, where g denotes the genus of M . The purpose of this note is to improve this theorem and obtain the best constant. The following is our main result:

Theorem. *Let M be a compact minimally immersed surface in the unit sphere S^3 . Then $\|S\|_2 \geq 2\sqrt{2}g\pi$. The equality sign holds if and only if M is either the equatorial sphere or the Clifford torus. In particular, if $\|S\|_2 \leq 2\sqrt{2}\pi$, then M is either the equatorial sphere or the Clifford torus.*

1. NOTATIONS AND AUXILIARY RESULTS

Let M be a compact connected minimally immersed surface in the unit sphere S^3 . Following the notations of [1], denote by $h = (h_{ij})$ the second fundamental form of M , and by S the square of the length of h , $S = \sum h_{ij}^2$. We need the following auxiliary results.

Lemma 1 [1]. $\frac{1}{2}\Delta S = S(2-S) + \sum h_{ijk}^2$, where h_{ijk} denote the covariant derivatives of h_{ij} .

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Lemma 2 [3]. *The set of all zeros of S is either the whole space M or at most a finite set of points.*

Lemma 3. $|\nabla S|^2 = 2S \sum h_{ijk}^2$.

From Lemmas 1 and 3, we see that if S is constant, then either $S \equiv 0$ or $S \equiv 2$.

Lemma 4. *If $g \geq 1$, then*

$$\lim_{\varepsilon \rightarrow 0} \sum_{i=1}^k \int_{\partial B\varepsilon(p_i)} \frac{S_r}{S} = 16\pi(g-1)$$

where $p_1, p_2, \dots, p_k$ constitute all the zeros of S and S_r denotes the derivative of S on $\partial B\varepsilon(p_i)$ in the radial direction from p_i .

Proof. At the points where S is positive, by Lemma 3, we get

$$(1) \quad \Delta \log S = 2(2 - S).$$

Integrating (1) over $M\varepsilon = M \setminus \bigcup_{i=1}^k B\varepsilon(p_i)$, we get from the Gauss equation

$$(2) \quad 2K = 2 - S,$$

where K is the Gauss curvature of M , the assertion by Stokes's theorem and the theorem of Gauss-Bonnet. $\square$

Lemma 5.

$$\int_M \sqrt{\frac{S}{2}} + \left(\frac{\pi}{4} - \sin^{-1} \sqrt{\frac{S}{2+S}} \right) \frac{2-S}{2} \geq (g+1)\pi^2.$$

Proof. Regard M as an immersed surface of $\mathbf{R}^4$. Then the total absolute curvature of M , in the sense of [2], is given by

$$\begin{aligned} & \int_M \int_0^{2\pi} \left| (\sin \theta)^2 - \frac{S}{2} (\cos \theta)^2 \right| d\theta dV \\ &= \int_M 2\sqrt{2S} + \left(\pi - 4 \sin^{-1} \sqrt{\frac{S}{2+S}} \right) \frac{2-S}{2}. \end{aligned}$$

By a well-known inequality of Chern-Lashof [2], we have

$$\int_M \sqrt{\frac{S}{2}} + \left(\frac{\pi}{4} - \sin^{-1} \sqrt{\frac{S}{2+S}} \right) \frac{2-S}{2} \geq \frac{\pi^2}{2} (b_0 + b_1 + b_2),$$

where b_i denotes the i th Betti number relative to the real field, for $i = 0, 1, 2$. Since M is of two-dimensional, $b_0 = 1$, $b_1 = 2g$, and $b_2 = 1$. $\square$

2. PROOF OF THEOREM

We may assume that S is positive except possibly at a finite set of points (Lemma 2). By using (1) and Lemmas 3 and 4, we get

$$\begin{aligned}
 & \int_M 1 + \frac{1}{4} h_{ijk}^2 - \sqrt{\frac{S}{2}} - \left(\frac{\pi}{4} - \sin^{-1} \sqrt{\frac{S}{2+S}} \right) \frac{2-S}{2} \\
 &= \int_M \left[\frac{1}{\sqrt{2}(\sqrt{2} + \sqrt{S})} - \frac{1}{2} \left(\frac{\pi}{4} - \sin^{-1} \sqrt{\frac{S}{2+S}} \right) \right] (2-S) + \frac{1}{4} h_{ijk}^2 \\
 &= \lim_{\varepsilon \rightarrow 0} \int_{M_\varepsilon} \frac{1}{2} \left[\frac{1}{\sqrt{2}(\sqrt{2} + \sqrt{S})} - \frac{1}{2} \left(\frac{\pi}{4} - \sin^{-1} \sqrt{\frac{S}{2+S}} \right) \right] \Delta \log S + \frac{|\nabla S|^2}{8S} \\
 &= \lim_{\varepsilon \rightarrow 0} \int_{M_\varepsilon} \frac{1}{2} \nabla \left[\frac{1}{2} \left(\frac{\pi}{4} - \sin^{-1} \sqrt{\frac{S}{2+S}} \right) - \frac{1}{\sqrt{2}(\sqrt{2} + \sqrt{S})} \right] \nabla \log S + \frac{|\nabla S|^2}{8S} \\
 &\quad - \lim_{\varepsilon \rightarrow 0} \int_{\partial M_\varepsilon} \frac{1}{2} \left[\frac{1}{\sqrt{2}(\sqrt{2} + \sqrt{S})} - \frac{1}{2} \left(\frac{\pi}{4} - \sin^{-1} \sqrt{\frac{S}{2+S}} \right) \right] \frac{S_r}{S} \\
 &= -(4-\pi)\pi(g-1) + \lim_{\varepsilon \rightarrow 0} \int_{M_\varepsilon} \left(\frac{1}{4} - \frac{1}{(2+S)(\sqrt{2} + \sqrt{S})^2} \right) \frac{|\nabla S|^2}{2S} \\
 &= -(4-\pi)\pi(g-1) + \int_M \left(\frac{1}{4} - \frac{1}{(2+S)(\sqrt{2} + \sqrt{S})^2} \right) h_{ijk}^2 \\
 &\geq -(4-\pi)\pi(g-1),
 \end{aligned}$$

where the equality sign holds if and only if S is constant. According to Lemma 5, we get

(3)

$$\begin{aligned}
 A + \frac{1}{4} \int_M h_{ijk}^2 &\geq -(4-\pi)\pi(g-1) + \int_M \sqrt{\frac{S}{2}} + \left(\frac{\pi}{4} - \sin^{-1} \sqrt{\frac{S}{2+S}} \right) \frac{2-S}{2} \\
 &\geq 2g\pi^2 - 4\pi(g-1),
 \end{aligned}$$

where A denotes the area of M . By combining (2) with the inequality (3), it follows that

$$\int_M 2S + h_{ijk}^2 = 4A + 16\pi(g-1) + \int_M h_{ijk}^2 \geq 8g\pi^2.$$

The desired inequality now follows from Lemma 1.

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DEPARTMENT OF APPLIED MATHEMATICS, NATIONAL CHIAO TUNG UNIVERSITY, HSINCHU, TAIWAN, REPUBLIC OF CHINA