ON WEAK CONTAINMENT PROPERTIES

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ABSTRACT. We prove, that two concepts of weak containment do not coincide, contradicting results in [1, Lemma 3.3 and Proposition 3.4]. The statement of Theorem 3.5 remains valid. There exist infinite tall compact groups G (i.e. the set $\{\sigma \in \hat{G}, \dim \sigma = n\}$ is finite for each positive integer n) having the mean-zero weak containment property. Such groups do not have the dual Bohr approximation property or $AP(\hat{G}) \neq C_\delta^*(G)$.

Let G be a compact group, then G is said to have the mean-zero weak containment property if there exists a net

$$\{g_\alpha\} \text{ in } L_0^2(G) = \left\{ f \in L^2(G), \int f(x)dx = 0 \right\}$$

such that $\|g_\alpha\|_2 = 1$ and $\lim_\alpha \|L_y g_\alpha - g_\alpha\|_2 = 0$ for all $y \in G$. L_y denotes the left translation operator, $L_y f(x) = f(y^{-1}x)$, see [4]. It was claimed in [1, Lemma 3.3], that this property were equivalent to the following weaker version: There exists a net $h_\alpha \in L_0^2(G)$, $\|h_\alpha\|_2 = 1$, such that $L_y h_\alpha - h_\alpha \rightarrow 0$ weakly in $L_0^2(G)$.

The statement of this lemma is false. We show, why the indicated proof does not work (the error in the proof of Lemma 3.3 [1] is not isolated in literature). Since for convex sets the weak closure coincides with the norm closure, we can replace (h_α) by some convex linear combinations g_α such that $\lim \|L_y g_\alpha - g_\alpha\|_2 = 0$, $\forall y \in G$. But unfortunately it may happen, that $\lim \|g_\alpha\|_2 = 0$.

In fact, if G is an infinite compact group, then $L_0^2(G)$ is infinite dimensional, therefore there exists a net $h_\alpha \in L_0^2(G)$, such that $\|h_\alpha\|_2 = 1$ and $\lim_\alpha \langle h_\alpha, f \rangle = 0$ for all $f \in L_0^2(G)$, consequently

$$\lim_\alpha \langle L_y h_\alpha - h_\alpha, f \rangle = \lim_\alpha \langle h_\alpha, L_{y^{-1}} f \rangle - \lim_\alpha \langle h_\alpha, f \rangle = 0$$

for all $f \in L_0^2(G)$.

This has been proved in [1, Proposition 3.4] for $G = \prod_{n=1}^\infty G_n$ a direct product of nontrivial compact groups.

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Remark 1. Example 1 shows that the mean-zero weak containment property does not hold for general infinite products of nontrivial compact groups as stated in [1, Proposition 3.4].

Example 1. If $n \geq 3$, $G = \prod_{p \in \mathbb{P}} Sl(n, F_p)$, $F_p = \mathbb{Z}/p\mathbb{Z}$, $\mathbb{P}$ denotes the set of all primes, then G does not have the mean-zero weak containment property.

Proof. (The sketch of proof given in [3] is incomplete.) The canonical homomorphisms $\varphi_p : \mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z} = F_p$, define a homomorphism $\varphi : Sl(n, \mathbb{Z}) \Rightarrow \prod_{p \in \mathbb{P}} Sl(n, F_p)$ ($n \geq 2$). φ is clearly injective (if $x \equiv y \pmod{p}$ for all $p \in \mathbb{P}$ then $x = y$). If we can show that the image of φ is dense in G , then G contains a dense subgroup isomorphic to $Sl(n, \mathbb{Z})$, i.e. a group satisfying Kazhdan's property T , if $n \geq 3$ and consequently G does not have the mean-zero weak containment property (cf. [5]). Therefore it suffices to prove

Lemma 1. Let E be a finite set of primes, $A_p = (a_{ij,p}) \in Sl(n, \mathbb{Z})$ such that $\det A_p \equiv 1 \pmod{p}$ for all $p \in E$, then there exists $A = (a_{ij}) \in Sl(n, \mathbb{Z})$ such that $a_{ij} \equiv a_{ij,p} \pmod{p}$ for all $p \in E$ and $1 \leq i, j \leq n$.

Proof. It suffices to consider the case $n = 2$, since $\prod_{p \in E} Sl(n, F_p)$ can be generated by nice subgroups $Sl(2, F_p)$, $p \in E$. Put $m = \prod_{p \in E} p$, then by the Chinese Remainder Theorem it follows that there exists $A = (a_{ij})$, $a_{ij} \equiv a_{ij,p} \pmod{p}$ for all $p \in E$, $1 \leq i, j \leq n$. We have $\det A \equiv \det A_p \equiv 1 \pmod{p}$ for all $p \in E$, consequently $\det A = 1 + mx$, $x \in \mathbb{Z}$.

In particular $(a_{12}, a_{22}, m) = 1$ (i.e. a_{12}, a_{22} and m have no nontrivial common divisors).

If $a_{i2} = d_i b_i$, $d_i = (a_{i2}, m)$, then $(d_1, d_2) = 1$. Setting $\bar{a}_{i2} = a_{i2} + x_i m$ ($\equiv a_{i2} \pmod{p}$ for all p) we may obtain that $(\bar{b}_1, \bar{b}_2) = 1$. (Dirichlet's prime number theorem implies even that we may assume that $\bar{b}_1, \bar{b}_2$ are distinct primes), or that $(\bar{a}_{12}, \bar{a}_{22}) = 1$. Therefore there exist $y_1, y_2 \in \mathbb{Z}$ such that $y_1 \bar{a}_{22} - y_2 \bar{a}_{12} = -x$. If $\bar{a}_{i1} = a_{i1} + y_i m$, $\bar{A} = (\bar{a}_{ij})$, then $\det \bar{A} = \det A + m(y_1 \bar{a}_{22} - y_2 \bar{a}_{12}) = 1 + mx - mx = 1$. This concludes the proof of the lemma. Considering the "root subgroups" U_{kl} instead of H_{uv} , the proof becomes easier, since in this case A itself can be chosen to be a transvection and consequently $\det A = 1$.

Remark 2. It can be easily seen that there always exist dense subgroups H of $G = \prod_{n=1}^{\infty} G_n$ (e.g. $H = \bigoplus G_n$ weak direct sum) and nets g_α such that $\|g_\alpha\|_2 = 1$ and $\lim_\alpha \|L_y g_\alpha - g_\alpha\|_2 = 0$, $\forall y \in H$.

Remark 3. $G = \prod_{n=1}^{\infty} S_n$, S_n the group of permutations of n elements, gives an example of a nonamenable group G having the mean-zero weak containment property, see [3, Example 5].

Remark 4. Let $d(H)$ denote the minimal number of generators of a finite group H . If $G = \prod_{n=1}^{\infty} H_n$, and $\limsup_{n \rightarrow \infty} d(H_n) = \infty$, then G has the mean-zero weak containment property. (Any finite set $\{y_1, y_2, \dots, y_k\}$ is contained in a subgroup $K = \prod_{n=1}^{\infty} K_n$, $K_n = H_n$ for $n \neq m$, $K_m = L_m \neq H_m$, if $d(H_m) > k$. If $g(x) = 1$, if $x \notin K$, $g(x) = 1 - 1/[G : K]$, if $x \in K$, then $g \in L_0^2(G)$, $\|g\|_2 \geq 1$ and $L_{y_i} g - g = 0$ for $1 \leq i \leq k$.) Therefore we are able to prove:

Proposition 1. There exist compact tall groups G having the mean-zero weak containment property.

If G_n is any sequence of pairwise nonisomorphic non-Abelian simple groups, then one can choose $G = \prod_{n=1}^{\infty} G_n^{k_n}$ for suitable k_n .

Proof. It follows from Hutchinson [2, Theorem 2.5] that any such groups are tall. Example 1 shows that in general one cannot take $k_n = 1$ for all n . I learned from Kowol that $d(G)$ is small for a lot of groups (e.g. $d(H) = 2$ for any simple non-Abelian group H). Clearly $d(H^r) > s$, if $H = \{h_1, h_2, \dots, h_m\}$, $r > m^s$, since $a_i = (a_{i1}, \dots, a_{ir})$, $1 \leq i \leq s$ cannot generate H^r ($a_{ij} = a_{ik}$, for some $j \neq k$ and $1 \leq i \leq s$). Therefore Proposition 1 easily follows from Remark 4.

Remark 5. Proposition 1 implies that there exist compact groups G which do not have the dual Bohr approximation property, see [1, Theorem 3.5].

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