UNIFORMLY CONTINUOUS AND WEAKLY ALMOST PERIODIC FUNCTIONS ON SOME TOPOLOGICAL SEMIGROUPS

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ABSTRACT. Let $LUC(S)$ be the space of bounded left uniformly continuous functions and $WAP(S)$ the space of bounded weakly almost periodic functions on a topological semigroup S . For a large class of topological semigroups called *stips* that are not relatively neocompact, we show that the quotient space $LUC(S)/WAP(S)$ contains an isometric linear copy of l^∞ . The result is particularly interesting on noting that the underlying topological semigroup is not assumed to support an absolutely continuous measure—a theme studied by the author in a previous paper.

Let S be a locally compact Hausdorff topological semigroup, $C(S)$ the space of all bounded real-valued continuous functions on S , $LUC(S) := \{f \in C(S) : \text{the map } x \rightarrow {}_x f \text{ of } S \text{ into } C(S) \text{ is norm continuous}\}$ the space of *left uniformly continuous functions* on S and $WAP(S) := \{f \in C(S) : \text{the set } \{{}_x f : x \in S\} \text{ is relatively weakly compact}\}$ the space of *weakly almost periodic functions* on S . (Here the function ${}_x f$ is given by ${}_x f(y) := f(xy)$ for all $f \in C(S)$ and $x, y \in S$.) The space of *right uniformly continuous functions*, $RUC(S)$, is the right-handed counterpart of $LUC(S)$ and the space of uniformly continuous functions on S is $UC(S) := LUC(S) \cap RUC(S)$. If we use the weak topology in place of norm topology on $C(S)$, we obtain the *weakly uniformly continuous functions* $WUC(S)$ in place of $US(S)$.

For any subsets A and B of S and points x in S , we write

$$\begin{aligned} AB &:= \{ab : a \in A \text{ and } b \in B\}, \\ A^{-1}B &:= \{y \in S : ay \in B \text{ for some } a \in A\}, \\ A^{-1}x &:= A^{-1}\{x\} \quad \text{and} \quad x^{-1}A := \{x\}^{-1}A. \end{aligned}$$

By symmetry the right-handed notation BA^{-1} , xA^{-1} , and Ax^{-1} can be easily defined. If $A_1, \dots, A_n$ are subsets of S , we define

$$\begin{aligned} A_1 \otimes A_2 &:= \{A_1 A_2, A_1^{-1} A_2, A_1 A_2^{-1}\} \\ A_1 \otimes A_2 \otimes A_3 &:= (U\{A_1 \otimes B : B \in A_2 \otimes A_3\}) \cup (U\{B \otimes A_3 : B \in A_1 \otimes A_2\}) \end{aligned}$$

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and hence inductively similarly define $A_1 \otimes A_2 \otimes \cdots \otimes A_n$.

Following [2, 4], a subset E of S is said to be *relatively neocompact* if E is contained in a union of sets in $A_1 \otimes \cdots \otimes A_n$ for some compact subsets $A_1, \dots, A_n$ of S .

We now define the underlying topological semigroup for our main result in this paper:

A locally compact topological semigroup S with an identity element 1 is called stip if for each neighborhood U of 1 we have that

$$\begin{aligned} x \in \text{int}(U^{-1}(Ux) \cap (xU)U^{-1}) \quad \text{for all } x \in S, \text{ and} \\ 1 \in \text{int}(U^{-1}u \cap vU^{-1}) \quad \text{for some } u, v \in U. \end{aligned}$$

For an extensive study of stips, we refer the reader to ([4, 6] and the references mentioned there). Stips form a very large class of topological semigroups many of which are known to support absolutely continuous measures (i.e. Radon measures on S continuous under translation by points of S)—see e.g. [4, 6].

In [3, 4] we showed that every nonrelatively neocompact topological semigroup S admitting an absolutely continuous measure is such that the quotient space $\text{WUC}(S)/\text{WAP}(S)$ contains an isometric linear copy of l^∞ (and so is nonseparable)—thus solving a problem raised earlier and partially studied for topological groups by E. E. Granirer [5] and Ching Chou [1]. We also showed that for a stip S , $\text{WAP}(S) \subseteq \text{UC}(S)$ and $\text{WUC}(S) = \text{UC}(S)$ and raised the conjecture that $\text{UC}(S)/\text{WAP}(S)$ contains an isometric copy of l^∞ whenever S is not relatively neocompact. (Here S is no longer assumed to support an absolutely continuous measure.) In this paper we solve the conjecture if the stip is assumed to be commutative or when $\text{LUC}(S)$ is used in place of $\text{UC}(S)$.

Theorem. *Let S be a stip that is not relatively neocompact. Then the quotient space $\text{LUC}(S)/\text{WAP}(S)$ contains an isometric linear copy of l^∞ .*

Proof. From the proof of [3, Theorem 7.3, p. 234] we can find compact neighborhoods V and W of the identity, 1 , $w \in W$ and sequences $\{x_n\}$, $\{y_m\}$ such that

$$(1) \quad \left. \begin{aligned} V &\subset W^{-1}w \cap wW^{-1} \cap W, \\ x &\in \text{int}(w^{-1}(Wx)) \quad \text{for all } x \in S \end{aligned} \right\}$$

and

$$(2) \quad \left. \begin{aligned} w^{-1}(Wx) \cap V^{-1}(Vx_n y_m) &\neq \emptyset \text{ implies} \\ \text{that } w^{-1}(Wx) \cap V^{-1}(Vx_i y_j) &= \emptyset \end{aligned} \right\}$$

if any one of the following items holds:

- (a) $n \leq m$ and $i > j$
- (b) $n > m$, $i > j$, and $n \neq i$
- (c) $n \leq m$, $i \leq j$, and $m \neq j$.

From [6, pp. 84–85] we can find a sequence of open neighborhoods of 1 , $\{V_k\}$, and a pseudometric ρ such that

$$\begin{aligned} V_{k+1}^2 &\subseteq V_k \subseteq V, \\ V_{k+1}^{-1}(V_{k+1}x) &\subseteq \{y \in S : \rho(y, x) \leq 2^{-k+1}\} \subseteq V_k^{-1}(V_k x), \\ \rho(xz, yz) &\leq \rho(x, y), \end{aligned}$$

for all $x, y, z \in S$ and $k \in \mathbb{N}$.

We thus have

$$(3) \quad \{y \in S : \rho(y, x) \leq 1\} \subseteq V^{-1}(Vx).$$

Let $h_{n,m}$ be the function on S given by

$$h_{n,m}(x) := 1 - \min(\rho(x, x_n y_m), 1).$$

By (3) we have that

$$(4) \quad \{x \in S : h_{n,m}(x) \neq 0\} \subseteq V^{-1}(Vx).$$

Let $A_k := \{x_{k_n}\}$ and $B_k := \{y_{k_m}\}$ be subsequences of $\{x_1, x_2, \dots\}$ and $\{y_1, y_2, \dots\}$, respectively, constructed as done in [3, p. 228] and consider the function

$$(5) \quad f_k := \sum_{n=1}^{\infty} \sum_{n \leq m} h_{k_n, k_m} - \sum_{m=1}^{\infty} \sum_{m < n} h_{k_n, k_m}.$$

From (2) and (4), we have that $0 \leq f_k \leq 1$.

Next we show that $f_k \in \text{LUC}(S)$. To the latter end, let $\varepsilon > 0$ be given. By the definition of a stip and the above choice of ρ , we can find an open neighborhood, 0 , of the identity such that

$$(6) \quad 0^{-1}(0x) \subseteq w^{-1}(Wx) \quad \text{and} \quad y \in 0^{-1}(0x) \quad \text{implies} \quad \rho(x, y) < \varepsilon$$

for all $x, y \in S$.

If both $f_k(x)$ and $f_k(y)$ equal zero, then $|f_k(x) - f_k(y)| < \varepsilon$. Now suppose $f_k(x) \neq 0$. Then $h_{k_i, k_j}(x) \neq 0$ for some $i, j \in \mathbb{N}$. By (4), (2), and the hypothesis on the choice of A_k and B_k , we have

$$h_{k_n, k_m}(x) = h_{k_n, k_m}(y) = 0 \quad \text{for those } (n, m) \text{ in (5) such that } (n, m) \neq (i, j).$$

Hence

$$\begin{aligned} |f_k(x) - f_k(y)| &= |h_{k_i, k_j}(x) - h_{k_i, k_j}(y)| \\ &= |\min(\rho(x, x_{k_i} y_{k_j}), 1) - \min(\rho(y, x_{k_i} y_{k_j}), 1)| \end{aligned}$$

We also note that

$$h_{k_i, k_j}(x) \neq 0 \quad \text{implies} \quad \rho(x, x_{k_i} y_{k_j}) < 1.$$

So

$$\begin{aligned} |f_k(x) - f_k(y)| &\leq \begin{cases} |\rho(x, x_{k_i} y_{k_j}) - \rho(y, x_{k_i} y_{k_j})| \leq \rho(x, y) < \varepsilon & \text{if } \rho(y, x_{k_i} y_{k_j}) \leq 1 \\ |\rho(x, x_{k_i} y_{k_j}) - (\rho(x, x_{k_i} y_{k_j}) + \varepsilon)| = \varepsilon & \text{if } \rho(y, x_{k_i} y_{k_j}) > 1, \end{cases} \end{aligned}$$

since $\rho(y, x_{k_i} y_{k_j}) \leq \rho(y, x) + \rho(x, x_{k_i} y_{k_j})$. Thus

$$y \in 0^{-1}(0x) \quad \text{implies} \quad |f_k(x) - f_k(y)| \leq \varepsilon \quad \text{for all } x, y \in S.$$

Since $ys \in 0^{-1}(0xs)$ whenever $y \in 0^{-1}(0x)$ for all x and y in S , we thus have that

$$y \in 0^{-1}(0x) \quad \text{implies} \quad \|_x(f_k) - _y(f_k)\| := \sup\{|f_k(xs) - f_k(ys)| : s \in S\} \leq \varepsilon.$$

Thus $f_k \in \text{LUC}(S)$.

By Grothendieck's characterisation of relatively weakly compact sets of functions, as similarly observed in [3], we have $f_k \notin \text{WAP}(S)$ since

$$f_k(x_{k_i}y_{k_j}) = \begin{cases} 1 & \text{if } i \leq j, \\ -1 & \text{if } i > j. \end{cases}$$

The rest of the proof now follows that of [3, §4] to show that for each $(c_k) \in l^\infty$

$$(c_k) \rightarrow \sum_{k=1}^{\infty} c_k f_k$$

yields an isometric linear copy of l^∞ into $\text{LUC}(S)/\text{WAP}(S)$.

Remarks. 1. If S is commutative then $\text{LUC}(S) = \text{UC}(S)$ and our theorem holds for uniformly continuous functions.

2. Let a stip S be such that $C^{-1}D$ and DC^{-1} are compact for all compact subsets C and D . Then the relatively neocompact subsets of S are precisely the relatively compact subsets. In particular if S is not compact then $\text{LUC}(S)/\text{WAP}(S)$ contains an isometric linear copy of l^∞ , by our theorem.

3. Relatively neocompact stips can be very large. In fact we may still have $\text{LUC}(S)/\text{WAP}(S)$ containing an isometric linear copy of l^∞ for certain relatively neocompact stips. For example

$$S := (\{0\} \times [0, \infty)) \cup \{(1, 0)\} \quad \text{with additive operation on } \{0\} \times [0, \infty)$$

and

$$x(1, 0) = (1, 0)x := (1, 0) \quad \text{for all } x \in S,$$

is a relatively neocompact stip. In fact $S = t^{-1}t$ where $t := (1, 0)$. But $H := \{0\} \times [0, \infty)$ is a clopen nonrelatively neocompact stip contained in S . Hence our theorem can be applied to H and trivially extend the result to S .

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