A STUDY OF ALMOST-EVERYWHERE SINGLETON-VALUED FILIPPOVS

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ABSTRACT. Filippov solutions of $x'(t) = f(t, x(t))$ are defined in terms of a certain set-valued function called the Filippov of f . Under some fairly general assumptions, we answer the following question: The Filippovs of f and g are equal if and only if f and g relate in what way?

1. INTRODUCTION

Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be given. We define the *Filippov* of f as follows:

$$\mathcal{F}[f](x) = \bigcap_{\varepsilon > 0} \bigcap_{Z: m(Z)=0} \overline{\text{conv}} f(B_\varepsilon(x) \setminus Z),$$

where m denotes Lebesgue measure, $\overline{\text{conv}} A$ represents the closure of the convex hull of the set A and $B_\varepsilon(x)$ represents the open ball of radius ε about the point x . Note that the Filippov is a set-valued function. The Filippov is used in defining a generalized solution of the ordinary differential equation $x' = f(t, x)$, particularly in the case of f discontinuous in x (see, for example [1, 2, 3]).

In this paper we investigate the following question:

$\mathcal{F}[f] = \mathcal{F}[g]$ if and only if f and g relate in what way?

(It is not difficult to show that if $f(x) = g(x)$ a.e., then $\mathcal{F}[f] = \mathcal{F}[g]$, since the Filippov operator “ignores” sets of measure zero.) To this end, consider the following. It is known that there exists a Lebesgue measurable set $A \subseteq \mathbf{R}$ such that for all $x \in \mathbf{R}$ and $\varepsilon > 0$, we have $0 < m(A \cap B_\varepsilon(x))/m(B_\varepsilon(x)) < 1$ (see, for example, [4]). Define $f, g: \mathbf{R} \rightarrow \mathbf{R}$ by $f(x) = \mathcal{H}_A(x)$ and $g(x) = \mathcal{H}_{A^c}(x)$ where $\mathcal{H}_A(x)$ and $\mathcal{H}_{A^c}(x)$ denote the characteristic functions of A and $\mathbf{R} \setminus A$ respectively. It can be shown that for each $x \in \mathbf{R}$, $\mathcal{F}[f](x) = \mathcal{F}[g](x) = [0, 1]$. However, f and g do not agree anywhere. Thus, it appears that, in general, there is no “nice” characterization of the relationship between f and g in the case in which $\mathcal{F}[f] = \mathcal{F}[g]$ (without repeating the definition in some trivial way). However, we show in this paper that under fairly general assumptions, a

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“nice” characterization *does* exist. Specifically, we obtain a characterization of the relationship between f and g in the case in which their Filippovs are singletons almost everywhere. We also give necessary and sufficient conditions for a function f to have a Filippov that satisfies this condition, plus necessary and sufficient conditions for f to have a Filippov with other properties of interest. In §2 we present some technical lemmas. In §3 we present and prove our main results.

2. TECHNICAL LEMMAS

This section consists of a list of lemmas and their proofs, which are used to obtain the main results in the final section. We assume throughout that $f, g: \mathbf{R} \rightarrow \mathbf{R}$ are Lebesgue measurable.

Lemma 1. *Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be given. If there exists a set $A \subseteq \mathbf{R}$ of full measure such that $f|_A$ is continuous, then for each $x \in A$ we have $\mathcal{F}[f](x) = \{f(x)\}$.*

Proof.

Claim. For each $x \in A$ we have $f(x) \in \mathcal{F}[f](x)$.

Proof of Claim. Fix $\varepsilon > 0$ and $Z \subseteq \mathbf{R}$ with $m(Z) = 0$. Since A has full measure, then for each $n \in \mathbf{N}$, $A \cap (B_{1/n}(x) \setminus Z)$ has full measure in $B_{1/n}(x)$. Thus for any n , there exists $x_n \in A \cap (B_{1/n}(x) \setminus Z)$. Clearly $x_n \rightarrow x$ and the continuity of f relative to A yields that $f(x_n) \rightarrow f(x)$. Thus $f(x) \in \text{cl } f(B_\varepsilon(x) \setminus Z) \subseteq \overline{\text{conv}} f(B_\varepsilon(x) \setminus Z)$. Since the above statements hold for any $\varepsilon > 0$ and $Z \subseteq \mathbf{R}$ of measure zero, we have $f(x) \in \bigcap_{\varepsilon > 0} \bigcap_{Z: m(Z)=0} \overline{\text{conv}} f(B_\varepsilon(x) \setminus Z) = \mathcal{F}[f](x)$ so the claim is established.

Assume there exists an $\bar{x} \in A$ such that $\mathcal{F}[f](\bar{x}) \neq \{f(\bar{x})\}$. Thus, there exists $y \in \mathbf{R}$ such that $y \neq f(\bar{x})$ and $y \in \mathcal{F}[f](\bar{x})$. (Note: we know $\mathcal{F}[f](\bar{x})$ is nonempty because of the claim.) Without loss of generality, assume that $y > f(\bar{x})$. For each $\varepsilon > 0$ and $Z \subseteq \mathbf{R}$ with $m(Z) = 0$, we have that $y \in \overline{\text{conv}} f(B_\varepsilon(\bar{x}) \setminus Z)$. Let $Z_0 = \mathbf{R} \setminus A$. Then for each $\varepsilon > 0$, $y \in \overline{\text{conv}} f(B_\varepsilon(\bar{x}) \setminus Z_0)$. Thus, for every $\varepsilon > 0$ there exist l_ε and u_ε such that $l_\varepsilon \leq y \leq u_\varepsilon$ and $l_\varepsilon, u_\varepsilon \in f(B_\varepsilon(\bar{x}) \setminus Z_0)$. For each $\varepsilon > 0$ let $w_\varepsilon \in B_\varepsilon(\bar{x}) \setminus Z_0 = B_\varepsilon(\bar{x}) \cap A$ satisfy $f(w_\varepsilon) = u_\varepsilon$. Let $\beta = y - f(\bar{x})$. We have thus shown that there exists $\beta > 0$ such that for each $\varepsilon > 0$ there exists $w_\varepsilon \in B_\varepsilon(\bar{x}) \cap A$ with $f(w_\varepsilon) - f(\bar{x}) = u_\varepsilon - f(\bar{x}) \geq y - f(\bar{x}) = \beta$. This contradicts the continuity of $f|_A$ at $\bar{x}$. $\square$

Lemma 2. *Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be given. Then $\mathcal{F}[f]$ is continuous when restricted to the set A on which it is a singleton (treated as a real-valued function).*

Proof. Assume not. Then there exists some $\bar{x} \in A$ at which $\mathcal{F}[f]$ is discontinuous. Thus, there exists $\beta > 0$ such that at least one of the following holds:

1. For every $\varepsilon > 0$, there exists $y_\varepsilon \in A$ with $|\bar{x} - y_\varepsilon| < \varepsilon$ but $\mathcal{F}[f](y_\varepsilon) - \mathcal{F}[f](\bar{x}) \geq \beta$.
2. For every $\varepsilon > 0$, there exists $y_\varepsilon \in A$ with $|\bar{x} - y_\varepsilon| < \varepsilon$ but $\mathcal{F}[f](y_\varepsilon) - \mathcal{F}[f](\bar{x}) \leq -\beta$.

Without loss of generality, assume that 1 holds. Choose $\varepsilon > 0$ and take a corresponding $y_\varepsilon \in A$. We know for every $\gamma > 0$ and every $Z \subseteq \mathbf{R}$ with $m(Z) = 0$, that $\mathcal{F}[f](y_\varepsilon) \in \overline{\text{conv}} f(B_\gamma(y_\varepsilon) \setminus Z)$. Choose $\gamma > 0$ so that $B_\gamma(y_\varepsilon) \subseteq B_\varepsilon(\bar{x})$. We then have for each $Z \subseteq \mathbf{R}$ of measure zero that $B_\gamma(y_\varepsilon) \setminus Z \subseteq B_\varepsilon(\bar{x}) \setminus Z$ and

so $f(B_\gamma(y_\varepsilon) \setminus Z) \subseteq f(B_\varepsilon(\bar{x}) \setminus Z)$. Thus we have $\mathcal{F}[f](y_\varepsilon) \in \overline{\text{conv}} f(B_\gamma(y_\varepsilon) \setminus Z) \subseteq \overline{\text{conv}} f(B_\varepsilon(\bar{x}) \setminus Z)$. That is, we have shown that for every $\varepsilon > 0$ and $Z \subseteq \mathbf{R}$ with $m(Z) = 0$, $\overline{\text{conv}} f(B_\varepsilon(\bar{x}) \setminus Z) \cap \{y : y \geq \beta + \mathcal{F}[f](\bar{x})\} \neq \emptyset$. Thus for every $\varepsilon > 0$ and every $Z \subseteq \mathbf{R}$ with $m(Z) = 0$, we have $[\mathcal{F}[f](\bar{x}), \mathcal{F}[f](\bar{x}) + \beta] \subseteq \overline{\text{conv}} f(B_\varepsilon(\bar{x}) \setminus Z)$, and thus

$$[\mathcal{F}[f](\bar{x}), \mathcal{F}[f](\bar{x}) + \beta] \subseteq \bigcap_{\varepsilon > 0} \bigcap_{Z : m(Z)=0} \overline{\text{conv}} f(B_\varepsilon(\bar{x}) \setminus Z) = \mathcal{F}[f](\bar{x}),$$

which is a contradiction. $\square$

Lemma 3. Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be given. If $\mathcal{F}[f]$ is a singleton almost everywhere, then $\mathcal{F}[f](x) = \{f(x)\}$ almost everywhere.

Proof. Assume the opposite. Then without loss of generality, it follows that there exists an $A \subseteq \mathbf{R}$ such that $m(A) > 0$, $\mathcal{F}[f](x)$ is a singleton for every $x \in A$ and $\mathcal{F}[f](x) < f(x)$ for every $x \in A$.

Claim 1. There exists $\delta > 0$ such that $m\{x \in A \mid f(x) > \mathcal{F}[f](x) + \delta\} > 0$.

Proof of Claim 1. Assume the opposite. Then for $\delta = 1/n$, $n = 1, 2, \dots$, $m\{x \in A \mid f(x) > \mathcal{F}[f](x) + 1/n\} = 0$. But

$$A = \bigcup_{n=1}^{\infty} \{x \in A \mid f(x) > \mathcal{F}[f](x) + 1/n\},$$

a countable union of sets of measure zero, which implies $m(A) = 0$, a contradiction.

Now let y be a point of density of $B \equiv \{x \in A \mid f(x) > \mathcal{F}[f](x) + \delta\}$ such that $y \in B$.

Claim 2. There exists $\bar{\varepsilon} > 0$ such that

$$\inf_{x \in B_{\bar{\varepsilon}}(y) \cap A} \{\mathcal{F}[f](x) + \delta - \mathcal{F}[f](y)\} > 0.$$

Proof of Claim 2. Define $g: C \rightarrow \mathbf{R}$ by $g(x) = \mathcal{F}[f](x) + \delta - \mathcal{F}[f](y)$ where C is the set on which $\mathcal{F}[f]$ is a singleton. By Lemma 2, $\mathcal{F}[f]$ is continuous restricted to the set C , hence g is continuous. The fact that $g(y) = \delta > 0$ and the continuity of g restricted to C implies that there exists $\bar{\varepsilon} > 0$ such that $\inf_{x \in B_{\bar{\varepsilon}}(y) \cap C} \{\mathcal{F}[f](x) + \delta - \mathcal{F}[f](y)\} > 0$. Noting that $A \subseteq C$, the claim follows.

Because y is a point of density of B , we know that for all $\varepsilon > 0$,

$$m[B_\varepsilon(y) \cap \{x \in A \mid f(x) > \mathcal{F}[f](x) + \delta\}] > 0.$$

Let

$$p = \inf_{x \in B_{\bar{\varepsilon}}(y) \cap A} \{\mathcal{F}[f](x) + \delta - \mathcal{F}[f](y)\}.$$

Claim 3. For all $\varepsilon > 0$, $m[B_\varepsilon(y) \cap \{x \in A \mid f(x) > \mathcal{F}[f](y) + p\}] > 0$.

Proof of Claim 3. Suppose the opposite. Then, there exists $\hat{\varepsilon} > 0$ such that $m[B_{\hat{\varepsilon}}(y) \cap \{x \in A \mid f(x) > \mathcal{F}[f](y) + p\}] = 0$. Choose $\varepsilon = \min\{\hat{\varepsilon}, \bar{\varepsilon}\}$ (where $\bar{\varepsilon}$ is chosen as in Claim 2). Then, $B_\varepsilon(y) \cap \{x \in A \mid f(x) > \mathcal{F}[f](y) + p\} \supseteq B_{\hat{\varepsilon}}(y) \cap \{x \in A \mid f(x) > \mathcal{F}[f](y) + p\} \supseteq B_{\hat{\varepsilon}}(y) \cap \{x \in A \mid f(x) > \mathcal{F}[f](x) + \delta\}$

by the previous claim. This last set has positive measure (by Claim 1), which yields a contradiction.

Thus for each $\varepsilon > 0$ and for every $Z \subseteq \mathbf{R}$ with $m(Z) = 0$, we have $\overline{\text{conv}}f(B_\varepsilon(y) \setminus Z) \supseteq [\mathcal{F}[f](y), \mathcal{F}[f](y) + p]$ and thus $[\mathcal{F}[f](y), \mathcal{F}[f](y) + p] \subseteq \mathcal{F}[f](y)$, which is a contradiction. $\square$

3. THE MAIN RESULTS

We now present and prove the main results of this paper. We first present theorems that characterize f , depending on the properties of its Filippov. The first result characterizes f in the case in which it is its own Filippov.

Theorem 1. *Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be given. For each $x \in \mathbf{R}$, $\mathcal{F}[f](x) = \{f(x)\}$ if and only if f is continuous.*

Proof. $(\Rightarrow)$: By Lemma 2, $\mathcal{F}[f]$ is continuous. Thus f , being the same function, has the same property.

$(\Leftarrow)$: This follows from Lemma 1, letting $A = \mathbf{R}$. $\square$

The following result characterizes f in the case in which its Filippov is always a singleton (not necessarily $\{f\}$).

Theorem 2. *Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be given. For each $x \in \mathbf{R}$, $\mathcal{F}[f](x)$ is a singleton if and only if f agrees a.e. with a continuous function $g: \mathbf{R} \rightarrow \mathbf{R}$.*

Proof. $(\Rightarrow)$: We need to construct a $g: \mathbf{R} \rightarrow \mathbf{R}$ with two properties:

- (a) g is a continuous, and
- (b) $f = g$ almost everywhere.

Define g to be the function that satisfies $\mathcal{F}[f](x) = \{g(x)\}$ for each $x \in \mathbf{R}$. The (a) requirement follows from Lemma 2, since $\mathcal{F}[f](x)$ is a singleton for each $x \in \mathbf{R}$. The (b) requirement follows from Lemma 3. It tells us that $\mathcal{F}[f](x) = \{f(x)\}$ a.e. and thus $f(x) = g(x)$ a.e.

$(\Leftarrow)$: Let $g: \mathbf{R} \rightarrow \mathbf{R}$ represent a function that is continuous and satisfies $g(x) = f(x)$ a.e. It follows easily that $\mathcal{F}[g](x) = \mathcal{F}[f](x)$ for all $x \in \mathbf{R}$. By Lemma 1, $\mathcal{F}[g](x) = \{g(x)\}$ for all $x \in \mathbf{R}$. Thus, for any $x \in \mathbf{R}$, $\mathcal{F}[f](x) = \mathcal{F}[g](x) = \{g(x)\}$, a singleton. $\square$

We now characterize f in the case in which its Filippov is a singleton almost everywhere. As a “bonus,” we get the equivalence of these two conditions with a third property, which is condition (ii) mentioned below.

Theorem 3. *Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be given. The following three properties are equivalent:*

- (i) $\mathcal{F}[f](x)$ is a singleton almost everywhere.
- (ii) $\mathcal{F}[f](x) = \{f(x)\}$ almost everywhere.
- (iii) There exists a set $A \subseteq \mathbf{R}$ of full measure such that $f|_A$ is continuous.

Proof. (i) $\Rightarrow$ (ii): This is precisely Lemma 3.

(ii) $\Rightarrow$ (iii): $\mathcal{F}[f](x)$ is a singleton a.e., hence, by Lemma 2, $\mathcal{F}[f]$ is continuous when restricted to a set of full measure, call it S_1 . By hypothesis, $\mathcal{F}[f](x) = \{f(x)\}$ on a set of full measure, call it S_2 . We now let $A = S_1 \cap S_2$, and note that $f|_A$ is continuous.

(iii) $\Rightarrow$ (ii): Follows easily from Lemma 1.

(ii) $\Rightarrow$ (i): Trivial, since, for each $x \in \mathbf{R}$, $\{f(x)\}$ is a singleton. $\square$

Remark. Clearly, the properties listed in Theorem 2 imply those of Theorem 3. However, the converse is not true, since it is easy to construct an f with a Filippov, which is a singleton only almost everywhere, say, $f: \mathbf{R} \rightarrow \mathbf{R}$ given by $f(x) = 1$ for $x < 0$ and $f(x) = -1$ for $x \geq 0$.

We now consider the question posed in the introduction: $\mathcal{F}[f] = \mathcal{F}[g]$ if and only if f and g relate in what way? The following theorem covers the case in which the Filippovs are singletons almost everywhere. Note also that we include a third property, which tells us that in the aforementioned case, it is impossible for the Filippovs to agree on a set of full measure, but disagree on a set of zero measure (except, of course, for the null set).

Theorem 4. *Let $f, g: \mathbf{R} \rightarrow \mathbf{R}$ be given such that both $\mathcal{F}[f](x)$ and $\mathcal{F}[g](x)$ are singletons a.e. Then, the following three properties are equivalent:*

- (i) *For each $x \in \mathbf{R}$, $\mathcal{F}[f](x) = \mathcal{F}[g](x)$.*
- (ii) *$\mathcal{F}[f](x) = \mathcal{F}[g](x)$ almost everywhere.*
- (iii) *$f(x) = g(x)$ almost everywhere.*

Proof. (i) $\Rightarrow$ (ii): Trivial.

(ii) $\Rightarrow$ (iii): Since $\mathcal{F}[f](x)$ is a singleton a.e., we have by Lemma 3 that $\mathcal{F}[f](x) = \{f(x)\}$ a.e. Similarly, $\mathcal{F}[g](x) = \{g(x)\}$ a.e. Thus, by hypothesis, $\{f(x)\} = \{g(x)\}$ a.e., that is $f(x) = g(x)$ a.e.

(iii) $\Rightarrow$ (i): Clear, since the construction of the Filippov “ignores” sets of measure zero. $\square$

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