PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 114, Number 2, February 1992

A DISCONJUGACY CRITERION OF W. T. REID FOR DIFFERENCE EQUATIONS

ALLAN PETERSON AND JERRY RIDENHOUR

(Communicated by Kenneth R. Meyer)

ABSTRACT. Our main result is a disconjugacy criterion for the selfadjoint vector difference equation $Ly(t) \equiv \Delta[P(t-1)\Delta y(t-1)] + Q(t)y(t) = 0$. This result is the analogue of a famous result of W. T. Reid for the corresponding differential equations case. Unlike the differential equations case we will see there is an exceptional case in which, as we will show by counterexample, the conclusion of the main result is no longer valid. A disfocality criterion is also given. We believe these results are new even in the scalar case.

We are concerned with the n-dimensional second order selfadjoint vector difference equation

$$\mathbf{L}y(t) \equiv \Delta[P(t-1)\Delta y(t-1)] + Q(t)y(t) = 0,$$

where $P(t)$ is an $n \times n$ Hermitian matrix function on the integer interval $[a, b+1] \equiv \{a, a+1, \dots, b+1\}$ with $P(t) > 0$ (positive definite) in $[a, b+1]$ and $Q(t)$ is an $n \times n$ Hermitian matrix function on $[a+1, b+1]$. Solutions of the equation $\mathbf{L}y(t) = 0$ are defined on the integer interval $[a, b+2]$. If $y(t)$ is a complex vector solution of $\mathbf{L}y(t) = 0$ on $[a, b+2]$, then

$$y^*(t)P(t-1)\Delta y(t-1) - \Delta y^*(t-1)P(t-1)y(t) = c$$

on $[a+1, b+2]$ for some constant c. If $c = 0$, we say $y(t)$ is a *prepared solution* of $\mathbf{L}y(t) = 0$. Hence, if $y(t)$ is a prepared solution of $\mathbf{L}y(t) = 0$, then

$$y^*(t)P(t-1)\Delta y(t-1) = \Delta y^*(t-1)P(t-1)y(t) \tag{1}$$

on $[a+1, b+2]$ (so $y^*(t)P(t-1)\Delta y(t-1)$ is real-valued on $[a+1, b+2]$). It follows from (1) that

$$y^*(t-1)P(t-1)y(t) = y^*(t)P(t-1)y(t-1) \tag{2}$$

on $[a+1, b+2]$ (so $y^*(t-1)P(t-1)y(t)$ is real-valued on $[a+1, b+2]$).

We now define what we mean by a *generalized zero* of a nontrivial prepared solution $y(t)$ of $\mathbf{L}y(t) = 0$. The definition is relative to the fixed interval $[a, b+2]$ and the left endpoint a is treated separately. In particular, we say $y(t)$ has a generalized zero at a if and only if $y(a) = 0$, while we say $y(t)$ has a

Received by the editors August 4, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 34C10; Secondary 34A30.
Key words and phrases. Disconjugate, prepared solution, generalized zero, disfocal.

generalized zero at $t_0 > a$ provided either $y(t_0) = 0$ or $y^*(t_0-1)P(t_0-1)y(t_0) \leq 0$ holds with $y(t_0-1) \neq 0$. In the real scalar case, this is the same definition used by Hartman [5]. For a discussion of this definition in more general situations, see [8, 9]. In [2, 3] Ahlbrandt and Hooker use different terminology but study what we call generalized zeros.

We say the equation $\mathbf{L}y(t) = 0$ is *disconjugate* on $[a, b+2]$ provided no nontrivial prepared solution has two generalized zeros in $[a, b+2]$. Our definition of disconjugacy is equivalent to that given by Ahlbrandt in [1]. Our main result is a disconjugacy criterion for $\mathbf{L}y(t) = 0$, which is an analogue of a result of W. T. Reid [11] for the corresponding differential equations case. Unlike the differential equations case, we see that there is an exceptional case in which, as we show by counterexample, the conclusion of the main result is no longer valid. We also give a disfocality criterion for $\mathbf{L}y(t) = 0$. Peil [6] has discussed disfocality for nth order linear difference equations. Peil and Peterson [7] discuss C-disfocality of $\mathbf{L}y(t) = 0$.

We now introduce notation that is used in the proof of the next theorem. Define a set $\mathscr{A}$ of admissible functions by

$$\mathscr{A} = \{\eta \colon [a, b+2] \to \mathbb{C}^n \text{ with } \eta(a) = 0 = \eta(b+2)\}.$$

Define a quadratic functional $\mathscr{J} : \mathscr{A} \to \mathbb{R}$ by

$$\mathscr{J}[\eta] = \sum_{t=a+1}^{b+2} \Delta\eta^*(t-1)P(t-1)\Delta\eta(t-1) - \sum_{t=a+1}^{b+1} \eta^*(t)Q(t)\eta(t).$$

Ahlbrandt and Hooker in [3] (see also [10]) have shown that $\mathbf{L}y(t) = 0$ is disconjugate on $[a, b+2]$ if and only if the quadratic form $\mathscr{J}$ is positive definite on $\mathscr{A}$. While our proofs are variational in nature, we do not use this result directly.

In the proof of Theorem 1, we use the fact [4, Lemma 11, p. 63] that if M and N are positive definite matrices then

$$M^{-1} + N^{-1} \geq 4[M+N]^{-1}. \tag{3}$$

As usual, we denote the $n \times n$ identity matrix by I.

Theorem 1. *Assume $q(t)$ is real-valued with $q(t) \geq 0$ and $Q(t) \leq q(t)I$ on $[a+1, b+1]$. Let*

$$D = 4\left\{\sum_{t=a}^{b+1} P^{-1}(t)\right\}^{-1} - \sum_{t=a+1}^{b+1} q(t)I.$$

Suppose either $D > 0$ or $D \geq 0$ and $\operatorname{card}\{t \in [a+1, b+1] : q(t) > 0\} \geq 2$ *holds. Then $\mathbf{L}y(t) = 0$ is disconjugate on $[a, b+2]$.*

Proof. Suppose, to the contrary, that $\mathbf{L}y(t) = 0$ is not disconjugate on $[a, b+2]$. Then there is a nontrivial prepared solution $y(t)$ of $\mathbf{L}y(t) = 0$ that has two generalized zeros in $[a, b+2]$. Let t_1 and t_2 with $t_1 < t_2$ be the first two generalized zeros of $y(t)$ in $[a, b+2]$. If $y(t_1) = 0$, let $c = t_1 + 1$ and $d = t_2$; if $y(t_1) \neq 0$, let $c = t_1$ and $d = t_2$. In either case, it is easy to see that $a+1 \leq c < d \leq b+2$, $y(c) \neq 0$, $y(d-1) \neq 0$,

$$y^*(c-1)P(c-1)y(c) \leq 0, \tag{4}$$

and

$$y^*(d-1)P(d-1)y(d) \leq 0. \tag{5}$$

Define η on $[a, b+2]$ by

$$\eta(t) = \begin{cases} y(t), & c \leq t \leq d-1, \\ 0, & \text{otherwise}. \end{cases}$$

Then $\eta(a) = 0 = \eta(b+2)$ so $\eta \in \mathscr{A}$. Since $\eta(c) = y(c) \neq 0$, η is nontrivial. Consider the case $c+1 < d$ (the degenerate case $c = d-1$ can be treated directly).

$$\begin{aligned}
\mathscr{J}[\eta] &= \sum_{t=a+1}^{b+2} \Delta\eta^*(t-1)P(t-1)\Delta\eta(t-1) - \sum_{t=a+1}^{b+1} \eta^*(t)Q(t)\eta(t) \\
&= \Delta\eta^*(c-1)P(c-1)\Delta\eta(c-1) + \sum_{t=c+1}^{d-1} \Delta y^*(t-1)P(t-1)\Delta y(t-1) \\
&\quad + \Delta\eta^*(d-1)P(d-1)\Delta\eta(d-1) - \sum_{t=c}^{d-1} y^*(t)Q(t)y(t).
\end{aligned}$$

Summing by parts, we obtain

$$\begin{aligned}
\mathscr{J}[\eta] &= y^*(c)P(c-1)y(c) + [y^*(t-1)P(t-1)\Delta y(t-1)]|_{c+1}^{d} \\
&\quad + \sum_{t=c+1}^{d-1} y^*(t)Q(t)y(t) + y^*(d-1)P(d-1)y(d-1) - \sum_{t=c}^{d-1} y^*(t)Q(t)y(t) \\
&= y^*(c)P(c-1)y(c) + y^*(d-1)P(d-1)\Delta y(d-1) - y^*(c)P(c)\Delta y(c) \\
&\quad + y^*(d-1)P(d-1)y(d-1) - y^*(c)Q(c)y(c) \\
&= y^*(c)[P(c-1)y(c) - P(c)\Delta y(c) - Q(c)y(c)] + y^*(d-1)P(d-1)y(d) \\
&= y^*(c)[\mathrm{L}y(c) + P(c-1)y(c-1)] + y^*(d-1)P(d-1)y(d) \\
&= y^*(c)P(c-1)y(c-1) + y^*(d-1)P(d-1)y(d) \\
&= y^*(c-1)P(c-1)y(c) + y^*(d-1)P(d-1)y(d).
\end{aligned}$$

We note that the above representation for $\mathscr{J}[\eta]$ is given in Lemma 3.3 of [3]; the proof is included here for completeness of presentation.

Therefore, by (4) and (5), we have that

$$\mathscr{J}[\eta] \leq 0. \tag{6}$$

Pick $e \in [c, d-1]$ so that

$$|y(e)| = \max_{t \in [c, d-1]} |y(t)| = \max_{t \in [c, d-1]} |\eta(t)|.$$

Now $y(e) \neq 0$ because $y(c) \neq 0$. Define $U(t)$ and $u(t)$, respectively, by

$$U(t) = \sum_{m=a}^{t-1} P^{-1}(m) \quad \text{and} \quad u(t) = U(t)U^{-1}(e)y(e)$$

for $a \leq t \leq e$, where $U(a) = 0$ by convention, Then

$$P(t-1)\Delta u(t-1) = U^{-1}(e)y(e) \tag{7}$$

for $a+1 \le t \le e$ with

$$u(a) = 0, \qquad u(e) = y(e) = \eta(e).$$

Because $P(t) > 0$ in $[a, b+1]$,

$$\begin{aligned}(8)\ 0 \le &\sum_{t=a+1}^{e} \Delta[\eta^*(t-1) - u^*(t-1)]P(t-1)\Delta[\eta(t-1) - u(t-1)] \\ = &\sum_{t=a+1}^{e} \Delta\eta^*(t-1)P(t-1)\Delta\eta(t-1) - \sum_{t=a+1}^{e} \Delta\eta^*(t-1)P(t-1)\Delta u(t-1) \\ &- \sum_{t=a+1}^{e} \Delta u^*(t-1)P(t-1)\Delta\eta(t-1) + \sum_{t=a+1}^{e} \Delta u^*(t-1)P(t-1)\Delta u(t-1).\end{aligned}$$

Using (7), we get

$$\begin{aligned}0 \le &\sum_{t=a+1}^{e} \Delta\eta^*(t-1)P(t-1)\Delta\eta(t-1) - \sum_{t=a+1}^{e} \Delta\eta^*(t-1)U^{-1}e(y)(e) \\ &- \sum_{t=a+1}^{e} y^*(e)U^{-1}(e)\Delta\eta(t-1) + \sum_{t=a+1}^{e} \Delta u^*(t-1)U^{-1}(e)y(e).\end{aligned}$$

It follows that

$$\begin{aligned}0 \le &\sum_{t=a+1}^{e} \Delta\eta^*(t-1)P(t-1)\Delta\eta(t-1) - \eta^*(e)U^{-1}(e)y(e) \\ &- y^*(e)U^{-1}(e)\eta(e) + y^*(e)U^{-1}(e)y(e).\end{aligned}$$

Because $\eta(e) = y(e)$, we obtain

$$(9) \qquad \sum_{t=a+1}^{e} \Delta\eta^*(t-1)P(t-1)\Delta\eta(t-1) \ge y^*(e)U^{-1}(e)y(e),$$

which by (8) is a strict inequality unless $\eta(t) = u(t)$ on $[a+1, e]$.

Next define $V(t)$ and $v(t)$, respectively, by

$$V(t) = -\sum_{m=t}^{b+1} P^{-1}(m) \quad \text{and} \quad v(t) = V(t)V^{-1}(e)y(e)$$

for $e \le t \le b+2$ (here $V(b+2) = 0$ is understood). Then

$$(10) \qquad P(t-1)\Delta v(t-1) = V^{-1}(e)y(e)$$

for $e+1 \le t \le b+2$ and

$$v(e) = y(e) = \eta(e), \qquad v(b+2) = 0.$$

Because $P(t) > 0$ on $[a, b+1]$,

(11)
$$\begin{aligned} 0 \le & \sum_{t=e+1}^{b+2} \Delta[\eta^*(t-1) - v^*(t-1)]P(1-t)\Delta[\eta(t-1) - v(t-1)] \\ = & \sum_{t=e+1}^{b+2} \Delta\eta^*(t-1)P(t-1)\Delta\eta(t-1) - \sum_{t=e+1}^{b+2} \Delta\eta^*(t-1)P(t-1)\Delta v(t-1) \\ & - \sum_{t=e+1}^{b+2} \Delta v^*(t-1)P(t-1)\Delta\eta(t-1) + \sum_{t=e+1}^{b+2} \Delta v^*(t-1)P(t-1)\Delta v(t-1). \end{aligned}$$

Hence, by (10),

$$\begin{aligned} 0 \le & \sum_{t=e+1}^{b+2} \Delta\eta^*(t-1)P(t-1)\Delta\eta(t-1) - \sum_{t=e+1}^{b+2} \Delta\eta^*(t-1)V^{-1}(e)y(e) \\ & - \sum_{t=e+1}^{b+2} y^*(e)V^{-1}(e)\Delta\eta(t-1) + \sum_{t=e+1}^{b+2} \Delta v^*(t-1)V^{-1}(e)y(e). \end{aligned}$$

It follows that

$$\begin{aligned} 0 \le & \sum_{t=e+1}^{b+2} \Delta\eta^*(t-1)P(t-1)\Delta\eta(t-1) + \eta^*(e)V^{-1}(e)y(e) \\ & + y^*(e)V^{-1}(e)\eta(e) - v^*(e)V^{-1}(e)y(e). \end{aligned}$$

Using $\eta(e) = v(e) = y(e)$, we get

$$\sum_{t=e+1}^{b+2} \Delta\eta^*(t-1)P(t-1)\Delta\eta(t-1) \ge -y^*(e)V^{-1}(e)y(e), \tag{12}$$

which by (11) is a strict inequality unless $\eta(t) = v(t)$ on $[e, b+2]$.

Combining (9) and (12), we have

$$\sum_{t=a+1}^{b+2} \Delta\eta^*(t-1)P(t-1)\Delta\eta(t-1) \ge y^*(e)[U^{-1}(e) - V^{-1}(e)]y(e) \tag{13}$$

with strict inequality unless

$$\eta(t) = \begin{cases} u(t), & a \le t \le e, \\ v(t), & e \le t \le b+2. \end{cases} \tag{14}$$

Using (3), we obtain

$$\sum_{t=a+1}^{b+2} \Delta\eta^*(t-1)P(t-1)\Delta\eta(t-1) \ge 4y^*(e)\left\{\sum_{t=a}^{b+1} P^{-1}(t)\right\}^{-1} y(e) \tag{15}$$

with strictly inequality unless (14) holds.

Next consider

$$\sum_{t=a+1}^{b+1} \eta^*(t)Q(t)\eta(t) \le \sum_{t=a+1}^{b+1} \eta^*(t)q(t)I\eta(t) \le y^*(e)\sum_{t=a+1}^{b+1} q(t)Iy(e). \tag{16}$$

Hence, by (15) and (16),

$$\mathcal{J}[\eta] \geq y^*(e)Dy(e), \tag{17}$$

where the inequality is strict unless (14) holds.

First assume $D > 0$. Then by (17), $\mathcal{J}[\eta] > 0$ and by (6), $\mathcal{J}[\eta] \leq 0$, which is impossible. Now assume $D \geq 0$ and $\operatorname{card}\{t \in [a+1, b+1] : q(t) > 0\} \geq 2$. If $c-1 > a$ then $u(c-1) = U(c-1)U^{-1}(e)y(e) \neq 0$ but $\eta(c-1) = 0$. Hence, (14) does not hold so the inequality in (17) is strict. This implies $\mathcal{J}[\eta] > 0$, contradicting (6). Hence, we must have that $c-1 = a$. Similarly, $d = b+2$. If

$$y^*(a)P(a)y(a+1) = y^*(c-1)P(c-1)y(c) < 0$$

or

$$y^*(d)P(d)y(d+1) = y^*(b+1)P(b+1)y(b+2) < 0,$$

then by (6), $\mathcal{J}[\eta] < 0$, which leads to a contradiction. Hence, we assume

$$y^*(a)P(a)y(a+1) = 0 = y^*(b+1)P(b+1)y(b+2).$$

If (14) does not hold, we get a contradiction. Consequently,

$$\eta(t) = \begin{cases} u(t), & a \leq t \leq e, \\ v(t), & e \leq t \leq b+2. \end{cases}$$

This implies

$$\begin{aligned} P(t-1)\Delta\eta(t-1) &= \begin{cases} P(t-1)\Delta u(t-1), & a+1 \leq t \leq e, \\ P(t-1)\Delta v(t-1), & e+1 \leq t \leq b+2; \end{cases} \\ &= \begin{cases} U^{-1}(e)y(e), & a+1 \leq t \leq e, \\ V^{-1}(e)y(e), & e+1 \leq t \leq b+2. \end{cases} \end{aligned}$$

Hence,

$$\Delta[P(t-1)\Delta\eta(t-1)]|_{t=e} = [V^{-1}(e) - U^{-1}(e)]y(e).$$

Because $y(t) - \eta(t)$ on $[a+1, b+1]$,

$$y^*(a)P(a)y(a+1) = y^*(a+1)P(a)y(a) = 0 = \eta^*(a+1)P(a)\eta(a)$$

and

$$y^*(b+1)P(b+1)y(b+2) = \eta^*(b+1)P(b+1)\eta(b+2),$$

we get

$$(\eta^*(t)\Delta[P(t-1)\Delta\eta(t-1)])|_{t=e} = (y^*(t)\Delta[P(t-1)\Delta y(t-1)])|_{t=e}.$$

Using $\mathbf{L}y(t) = 0$, we obtain

$$y^*(e)[V^{-1}(e) - U^{-1}(e)]y(e) = -y^*(e)Q(e)y(e).$$

It follows that

$$y^*(e)Sy(e) = 0, \tag{18}$$

where

$$S = U^{-1}(e) - V^{-1}(e) - Q(e).$$

Using (3) and $Q(t) \leq q(t)I$ on $[a+1, b+1]$, we obtain

$$S \geq 4\left\{\sum_{t=a}^{b+1} P^{-1}(t)\right\}^{-1} - q(e)I.$$

Because $\operatorname{card}\{t \in [a+1, b+1] : q(t) > 0\} \geq 2$ and $q(t) \geq 0$ in $[a+1, b+1]$, we have

$$S > 4\left\{\sum_{t=a}^{b+1} P^{-1}(t)\right\}^{-1} - \sum_{t=a+1}^{b+1} q(t)I = D \geq 0.$$

Because $S > 0$, it follows from (18) that $y(e) = 0$, which is our final contradiction. □

If $\operatorname{card}\{t \in [a+1, b+1] : q(t) > 0\} = 0$ in Theorem 1, then $Q(t) \leq 0$ on $[a+1, b+1]$, and it is well known that $\mathbf{L}y(t) = 0$ is disconjugate in this case. We now give an example where the cardinality of this set is one, $D = 0$, and all the other hypotheses of Theorem 1 are satisfied except that the difference equation is not disconjugate on $[a, b+2]$.

Example. Consider the vector difference equation

$$\Delta^2 y(t-1) + Q(t)y(t) = 0, \qquad a+1 \leq t \leq b+1, \tag{19}$$

where $b - a$ is an even integer and $Q(t) = q(t)I$ with $q(t)$ defined on $[a+1, b+1]$ by

$$q(t) = \begin{cases} 0, & t \neq (a+b+2)/2, \\ 4/(b+2-a), & t = (a+b+2)/2. \end{cases}$$

Note that

$$D = 4\{(b+2-a)I\}^{-1} - \frac{4}{b+2-a}I = 0$$

and

$$\operatorname{card}\{t \in [a+1, b+1] : q(t) > 0\} = 1.$$

Let $y(t)$ be the solution of (19) satisfying

$$y(a) = 0, \qquad y(a+1) = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

It is easy to see that $y(t)$ is a nontrivial prepared solution of (19) with $y(a) = y(b+2) = 0$. Hence, (19) is not disconjugate on $[a, b+2]$.

We say that $\mathbf{L}y(t) = 0$ is *right disfocal* on $[a, b+2]$ provided there is no nontrivial prepared solution $y(t)$ of $\mathbf{L}y(t) = 0$ and an integer $d \in [a+1, b+2]$ such that $\Delta y(d) = 0$ and y has a generalized zero in $[a, d]$. For results on right disfocality, see [6, 7]. In particular, our proof of Theorem 2 uses a quadratic form $\mathscr{J}[\eta]$, which for our application, agrees with one studied in [7].

We now give a sufficient condition for $\mathbf{L}y(t) = 0$ to be right disfocal on $[a, b+2]$. We believe this result is new, even in the scalar case.

Theorem 2. *Assume $q(t)$ is real-valued with $q(t) \geq 0$ and $Q(t) \leq q(t)I$ on $[a+1, b+1]$. Let*

$$F = \left\{\sum_{t=a}^{b} P^{-1}(t)\right\}^{-1} - \sum_{t=a+1}^{b+1} q(t)I.$$

If either $F > 0$ *or* $F \geq 0$ *and* $q(t) \not\equiv 0$ *on* $[a+1, b]$ *holds, then* $\mathrm{L}y(t) = 0$ *is right disfocal on* $[a, b+2]$.

Proof. Assume $\mathrm{L}y(t) = 0$ is not right disfocal on $[a, b+2]$. Then there is a nontrivial prepared solution $y(t)$ and an integer d in $[a+1, b+1]$ such that $\Delta y(d) = 0$ and $y(t)$ has as generalized zero in $[a, d]$. Let t_1 be the first generalized zero of $y(t)$ in $[a, d]$. If $y(t_1) = 0$, let $c = t_1 + 1$. If $y(t_1) \neq 0$, let $c = t_1$. In either case, we have $a + 1 \leq c \leq d$, $y(c) \neq 0$, and

$$y^*(c-1)P(c-1)y(c) \leq 0. \tag{20}$$

Define $\eta(t)$ on $[a, d+1]$ by

$$\eta(t) = \begin{cases} 0, & a \leq t \leq c-1, \\ y(t), & c \leq t \leq d+1. \end{cases}$$

Define $\mathscr{J}[\eta]$ by

$$\begin{aligned} \mathscr{J}[\eta] &= \sum_{t=a+1}^{d} \Delta\eta^*(t-1)P(t-1)\Delta\eta(t-1) - \sum_{t=a+1}^{d} \eta^*(t)Q(t)\eta(t) \\ &= \Delta\eta^*(c-1)P(c-1)\Delta\eta(c-1) + \sum_{t=c+1}^{d} \Delta y^*(t-1)P(t-1)\Delta y(t-1) \\ &\quad - \sum_{t=c}^{d} y^*(t)Q(t)y(t). \end{aligned}$$

Summing by parts and using the fact that $y(t)$ is a prepared solution, we obtain, by (20),

$$\begin{aligned} \mathscr{J}[\eta] &= y^*(c)P(c-1)y(c) + [y^*(t-1)P(t-1)\Delta y(t-1)]_{c+1}^{d+1} \\ &\quad + \sum_{t=c+1}^{d} y^*(t)Q(t)y(t) - \sum_{t=c}^{d} y^*(t)Q(t)y(t) \\ &= y^*(c)P(c-1)y(c) - y^*(c)P(c)\Delta y(c) - y^*(c)Q(c)y(c) \\ &= y^*(c)P(c-1)y(c-1) = y^*(c-1)P(c-1)y(c) \leq 0. \end{aligned} \tag{21}$$

We now set out to contradict (21). Pick $e \in [c, d]$ so that

$$|y(e)| = \max_{t \in [c, d]} |y(t)| = \max_{t \in [a, b+2]} |\eta(t)|.$$

Then set

$$u(t) = U(t)U^{-1}(e)y(e)$$

for $a \leq t \leq e$ where

$$U(t) = \sum_{m=a}^{t-1} P^{-1}(m).$$

It follows as in the proof of Theorem 1 that

$$P(t-1)\Delta u(t-1) = U^{-1}(e)y(e)$$

for $a + 1 \leq t \leq e$,

$$u(a) = 0, \qquad u(e) = y(e),$$

and

$$\sum_{t=a+1}^{e} \Delta\eta^*(t-1)P(t-1)\Delta\eta(t-1) \geq y^*(e)U^{-1}(e)y(e) \tag{22}$$

with equality if and only if $\eta(t) \equiv u(t)$ on $[a, e]$.

Next note that

$$\sum_{t=e+1}^{d} \Delta\eta^*(t-1)P(t-1)\Delta\eta(t-1) \geq 0 \tag{23}$$

with equality if and only if $\Delta\eta(t-1) = 0$ for $e+1 \leq t \leq d+1$ (which happens if and only if $\eta(t) \equiv y(t) \equiv y(d)$ for t in $[e+1, d]$).

Combining (22) and (23), we have

$$\sum_{t=a+1}^{d} \Delta\eta^*(t-1)P(t-1)\Delta\eta(t-1) \geq y^*(e)U^{-1}(e)y(e) \tag{24}$$

with equality if and only if

$$\eta(t) = \begin{cases} u(t), & a \leq t \leq e, \\ y(t) = y(d), & e+1 \leq t \leq d. \end{cases} \tag{25}$$

Furthermore,

$$\sum_{t=a+1}^{d} \eta^*(t)Q(t)\eta(t) \leq \sum_{t=a+1}^{d} y^*(e)q(t)y(e). \tag{26}$$

Hence, by (24) and (26),

$$\mathcal{J}[\eta] \geq y^*(e)Sy(e), \tag{27}$$

where

$$S = \left\{\sum_{t=a}^{e-1} P^{-1}(t)\right\}^{-1} - \sum_{t=a+1}^{d} q(t)I$$

and the inequality in (27) is strict if (25) does not hold.

Note that

$$S \geq \left\{\sum_{t=a}^{b} P^{-1}(t)\right\}^{-1} - \sum_{t=a+1}^{b+1} q(t)I = F \geq 0. \tag{28}$$

If $F > 0$ then $S > 0$, so by (27), $\mathcal{J}[\eta] > 0$, contradicting (21). Now assume $F \geq 0$. If (27) is a strict inequality, then we contradict (21). Hence, (25) must hold. If $c-1 > a$, then $u(c-1) \neq 0$ and $\eta(c-1) = 0$, which contradicts (25). Hence, $c-1 = a$ and we must have

$$\eta(t) = \begin{cases} u(t), & a \leq t \leq e, \\ y(d+1), & e \leq t \leq d+1. \end{cases}$$

If

$$y^*(a)P(a)y(a+1) = y^*(c)P(c)y(c+1) < 0,$$

then (21) is a strict inequality and we easily get a contradiction. Hence, we assume $y^*(a)P(a)y(a+1)=0$. Because

$$y^*(a+1)P(a)y(a)=y^*(a)P(a)y(a+1)=0,$$

we get

$$(y^*(t)\Delta[P(t-1)\Delta y(t-1)])|_{t=e}=-y^*(e)U^{-1}(e)y(e)=-y^*(e)Q(e)y(e).$$

This implies that

$$y^*(e)Hy(e)=0 \tag{29}$$

where

$$H=U^{-1}(e)-Q(e).$$

If $e<b+1$ then $H>F\geq 0$, and so by (29), $y(e)=0$, which is a contradiction. Now assume $e=b+1$. Then

$$H=U^{-1}(b+1)-Q(b+1)\geq\left\{\sum_{t=a}^{b}P^{-1}(t)\right\}-q(b+1)I.$$

Because $q(t)\not\equiv 0$ in $[a+1,b]$,

$$H>F\geq 0,$$

and we have a contradiction as before. □

References

1. C. Ahlbrandt, *Continued fraction representations of maximal and minimal solutions of a discrete matrix Riccati equation*, SIAM J. Math. Anal. (to appear).
2. C. Ahlbrandt and J. Hooker, *Recessive solutions of symmetric three term recurrence relations*, Oscillation, Bifurcation, and Chaos, Canad. Math. Soc. Conf. Proc., vol. 8 (F. Atkinson, W. Langford, and A. Mingarelli, eds.), 1987, pp. 3–42.
3. ——, *Riccati matrix difference equations and disconjugacy of discrete linear systems*, SIAM J. Math. Anal. **19** (1988), 1183–1197.
4. W. Coppel, *Disconjugacy*, Lecture Notes in Math., vol. 220, Springer-Verlag, Berlin, Heidelberg, and New York, 1971.
5. P. Hartman, *Difference equations: disconjugacy, principal solutions, Green's functions, complete monotonicity*, Trans. Amer. Math. Soc. **246** (1978), 1–30.
6. T. Peil, *Criteria for right disfocality of an nth order linear difference equation*, Rocky Mountain J. Math. (to appear).
7. T. Peil and A. Peterson, *Criteria for C-disfocality of a self-adjoint vector difference equation*, preprint.
8. A. Peterson and J. Ridenhour, *Atkinson's superlinear oscillation theorem for matrix difference equations*, SIAM J. Math. Anal. **22** (1991), 774–784.
9. ——, *Oscillation of second order linear matrix difference equations*, J. Differential Equations **89** (1991), 69–88.
10. ——, *Disconjugacy for a second order system of difference equations*, Differential Equations: Stability and Control (S. Elaydi, ed.), Lecture Notes in Pure and Appl. Math., vol. 127, Marcel Dekker, 1990, pp. 423–429.
11. W. Reid, *Oscillation criteria for linear differential systems with complex coefficients*, Pacific J. Math. **6** (1956), 733–751.

Department of Mathematics and Statistics, University of Nebraska-Lincoln, Lincoln, Nebraska 68588-0323

Department of Mathematics and Statistics, Utah State University, Logan, Utah 84322-3900