ON COEFFICIENT INEQUALITIES FOR MEROMORPHIC UNIVALENT FUNCTIONS

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ABSTRACT. We obtain some coefficient inequalities for the class Σ consisting of functions of the form $f(z) = z + b_0 + b_1/z + \cdots$ that are meromorphic and univalent in the exterior of the unit circle $|z| = 1$. These inequalities disprove two conjectures of Schober about linear functionals on Σ .

1. INTRODUCTION

Let Σ denote the class of functions of the form

$$(1) \quad f(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \frac{b_3}{z^3} + \cdots$$

that are meromorphic and univalent in $\Delta = \{z : |z| > 1\}$. Let L be a complex-valued continuous linear functional on the space of all meromorphic functions in Σ , under the topology of uniform convergence on compact subsets. Suppose that L is not constant on Σ and that L does not depend on the coefficient b_0 of f . We pose the extremal problem of maximizing $\operatorname{Re}\{L\}$ over Σ . The extremal functions are said to be support points of Σ associated with the linear functional L .

It is well known (see [1, Theorem 10.1], [2], or [3, Chapter 10]) that every support point $f \in \Sigma$ maps Δ onto the complement of a continuum Γ that consists of a finite number of analytic arcs satisfying the differential equation

$$(2) \quad L\left(\frac{1}{f-w}\right) dw^2 \geq 0.$$

Schober [4] posed the following two conjectures for general linear problems.

1. If f is a support point of Σ associated with a linear functional L , then $L(\frac{1}{f-w})$ as a function of w can have only simple zeros on the continuum Γ omitted by f .

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2. For each $k \geq 2$ there is a function $f \in \Sigma$ so that

$$\operatorname{Re} \left\{ b_{k+1} + \sum_{j=1}^k \frac{j}{k+1} \binom{2k+2}{j+k+1} b_j \right\} > \frac{1}{k+1} \binom{2k+2}{k}.$$

In particular, there exists a function in Σ for which

$$\operatorname{Re}\{b_3 + 4b_2 + 5b_1\} > 5.$$

In this paper we prove the coefficient inequalities

$$\operatorname{Re}\{b_3 + 4b_2 + 5b_1\} \leq 5, \quad \operatorname{Re}\{b_3 + b_2 + \tfrac{5}{4}b_1\} \leq \tfrac{5}{4},$$

and

$$\operatorname{Re}\{b_3 + b_1\} \leq 1,$$

with equalities holding only for

$$(3) \quad f(z) = z + 1/z + \text{const}.$$

The first inequality shows that conjecture 2 is not true for $k = 2$.

Functions $f_1(z) = z + 2 + 1/z$, $f_2(z) = z + 1 + 1/z$, and $f_3(z) = z + 1/z$ are support points of Σ associated with the linear functionals $L_1(f) = b_3 + 4b_2 + 5b_1$, $L_2(f) = b_3 + b_2 + \frac{5}{4}b_1$, and $L_3(f) = b_3 + b_1$ respectively. Now $L_j(\frac{1}{f_j - w}) = w^2$, $j = 1, 2, 3$, have double zero $w = 0$, which lies on $\Gamma_1 = \{w: 0 \leq w \leq 4\}$, $\Gamma_2 = \{w: -1 \leq w \leq 3\}$, and $\Gamma_3 = \{w: -2 \leq w \leq 2\}$ omitted, respectively, by f_1 , f_2 , and f_3 . Each of these three examples shows that conjecture 1 is not true.

2. SOME LEMMAS

In order to derive our results, we first prove three lemmas, which may further be used to derive other coefficient inequalities involving b_1 , b_2 , and b_3 .

Lemma 1. Let $f(z) = z + b_0 + b_1/z + \dots$ be a support point of Σ associated with the linear functional L for which

$$(4) \quad L\left(\frac{1}{f-w}\right) = w^2 + 1 - b_1.$$

Then the omitted set Γ of f cannot contain both roots of $w^2 = b_1 - 1$ unless f is the function (3).

Proof. The differential equation for Γ is now of the form

$$(5) \quad (w^2 + 1 - b_1) dw^2 \geq 0.$$

If we put $b_1 - 1 = \rho^2 e^{i\alpha}$, α real, then $\frac{1}{2}\pi \leq |\alpha| \leq \pi$ since $\operatorname{Re}\{b_1\} \leq 1$. Now equation (5) can be written as

$$(6) \quad e^{i2\alpha}(we^{-i\frac{1}{2}\alpha} + \rho)(we^{-i\frac{1}{2}\alpha} - \rho)(d(we^{-i\frac{1}{2}\alpha}))^2 \geq 0.$$

Notice that the Schwarz–Christoffel transformation

$$\zeta = \int_0^w (t + \rho)^{\frac{1}{2}}(t - \rho)^{\frac{1}{2}} dt$$

maps the upper halfplane $\text{Im}\{w\} > 0$ onto the exterior of an infinite semistrip of the form

$$\text{Re}\{\zeta\} > 0, \quad |\text{Im}\{\zeta\}| < h,$$

where

$$(7) \quad h = \text{Im} \left\{ \int_0^\rho (t + \rho)^{\frac{1}{2}} (t - \rho)^{\frac{1}{2}} dt \right\}.$$

Equation (6) merely states that Γ is the pre-image by the transformation

$$(8) \quad \zeta = \phi(w) = e^{i\alpha} \int_0^{we^{-i\frac{1}{2}\alpha}} (te^{-i\frac{1}{2}\alpha} + \rho)^{\frac{1}{2}} (te^{-i\frac{1}{2}\alpha} - \rho)^{\frac{1}{2}} d(te^{-i\frac{1}{2}\alpha})$$

of a line segment $\wedge$ in the ζ -plane parallel to the real axis $\text{Im}\{\zeta\} = 0$. Since the points $\rho e^{-i\alpha/2}$ and $-\rho e^{-i\alpha/2}$ are on Γ , the corresponding points $ihe^{i\alpha}$ and $-ihe^{-i\alpha}$ are on $\wedge$ so that $|\alpha| = \frac{1}{2}\pi$ and $\text{Re}\{b_1 - 1\} = 0$. Hence $b_1 = 1$ and the conclusion of Lemma 1 follows.

Lemma 2. *Under the hypotheses of Lemma 1, the omitted set Γ by f cannot omit both the points of $(b_1 - 1)^{1/2}$ and $-(b_1 - 1)^{1/2}$ unless f is the function (3).*

Proof. If we parametrize Γ by $w = f(e^{i\theta})$, then $dw/d\theta = izf'(z)$, $z = e^{i\theta}$, and (5) becomes

$$z^2 f'(z)^2 [f(z)^2 + 1 - b_1] \leq 0,$$

where $|z| = 1$. The left-hand side is an analytic function on Δ except for a single pole of order four at infinite. This function can be analytically continued into the interior of the unit circle by Schwarz reflection principle. In the neighborhood of infinity,

$$\begin{aligned} z^2 f'(z)^2 [f(z)^2 + 1 - b_1] &= z^4 + 2b_0 z^3 + (b_0^2 + 1 - b_1) z^2 - 2(b_2 + b_0 b_1) z \\ &\quad - 2(2b_3 + 3b_0 b_2 + b_0^2 b_1 + b_1) + \cdots. \end{aligned}$$

It has a pole of order four at $z = 0$. We may calculate it explicitly to obtain the differential equation

$$(9) \quad z^2 f'(z)^2 [f(z)^2 + 1 - b_1] = R(z),$$

where

$$\begin{aligned} (10) \quad R(z) &= z^4 + 2b_0 z^3 + (b_0^2 + 1 - b_1) z^2 - 2(b_2 + b_0 b_1) z \\ &\quad - 2(2b_3 + 3b_0 b_2 + b_0^2 b_1 + b_1) \\ &\quad - \frac{2(\bar{b}_2 + \bar{b}_0 \bar{b}_1)}{z} + \frac{\bar{b}_0^2 + 1 - \bar{b}_1}{z^2} + \frac{2\bar{b}_0}{z^3} + \frac{1}{z^4}. \end{aligned}$$

If $b_1 \neq 1$ then there are two distinct points z_1 and z_2 in Δ such that $f(z_1) = (b_1 - 1)^{1/2}$ and $f(z_2) = -(b_1 - 1)^{1/2}$. Then $z_1^* = z_1 |z_1|^{-2}$ and $z_2^* = z_2 |z_2|^{-2}$ as well as z_1 and z_2 are simple zeros of $R(z)$ by (9) and the fact that $R(z)$ is real on $|z| = 1$. Since $f'(z)$ is free of zeros outside and inside the unit circle $|z| = 1$, then either

$$(a) \quad f(z_1^*) = (b_1 - 1)^{1/2}, \quad f(z_2^*) = -(b_1 - 1)^{1/2} \quad \text{or}$$

$$(b) \quad f(z_1^*) = -(b_1 - 1)^{1/2}, \quad f(z_2^*) = (b_1 - 1)^{1/2}.$$

The composite function $\zeta = \phi(f(z))$, defined by (8), maps the circle $|z| = 1$

onto $\wedge$ lying on a horizontal line H in the ζ -plane. By Schwarz reflection principle, $\zeta_j = \phi(f(z_j))$ and $\zeta_j^* = \phi(f(z_j^*))$, $j = 1, 2$, are symmetric with respect to H . That is, $ihe^{i\alpha}$ and $-ihe^{i\alpha}$ are on H if (a) occurs and are symmetric with respect to H if (b) occurs, where h is given by (7).

In case (a) we must have $|\alpha| = \frac{1}{2}\pi$ and so $b_1 = 1$.

In case (b) we must have $\alpha = \pi$, by the previous choice of α , and H must be the real axis $\text{Im}\{\zeta\} = 0$. Thus $b_1 = 1$ again. The proof is complete.

Lemma 3. *Under the hypotheses of Lemma 1, if the omitted set Γ by f contains just one root of the equation $w^2 + 1 - b_1 = 0$, then b_1 is real and there exists two numbers r and η with $r > 1$, $|\eta| = 1$ such that*

$$(11) \quad \text{Im}\{(1 - 3r^2)\eta^2 - 2b_0r\eta + 2\bar{b}_0\bar{\eta} + 2r\bar{\eta}^2\} = 0,$$

$$(12) \quad 1 - b_1 = \text{Re}\{(1 - 3r^2)\eta^2 - 2b_0r\eta + 2\bar{b}_0\bar{\eta} + 2r\bar{\eta}^2\},$$

$$(13) \quad (1 - b_1) \log \frac{-(r-1)^2\eta^2}{1-b_1} + 2b_0^2 + 2b_0r\eta + 6\bar{b}_0\bar{\eta} + 4r\bar{\eta}^2 + 4b_1 = 0,$$

$$(14) \quad f(f^2 + 1 - b_1)^{\frac{1}{2}} + (1 - b_1) \log[f + (f^2 + 1 - b_1)^{\frac{1}{2}}] \\ = \left(z + r\eta + 2b_0 - \frac{r\bar{\eta}^2 + 2\bar{b}_0\bar{\eta}}{z} - \frac{\bar{\eta}}{z^2} \right) Q - \frac{1 - b_1}{2} \log \frac{-\eta^2}{1 - b_1} \\ + (1 - b_1) \log(Q + z - r\eta) - \log \left(1 - \frac{Q + \eta}{z} \right),$$

where $Q = (z^2 - 2r\eta z + \eta^2)^{1/2}$.

Proof. The left-hand side of (9) now has only one zero in Δ , say $\rho\eta$, $|\eta| = 1$. Hence $R(z)$ has two simple zeros $\rho\eta$ and η/ρ . $R(z)$ has three double roots on the circle $|z| = 1$ corresponding to the three tips of Γ . Therefore

$$(15) \quad R(z) = (z + \alpha + \bar{\alpha}\bar{\eta}/z + \bar{\eta}/z^2)^2(z^2 - 2r\eta z + \eta^2) \\ = z^4 + 2(\alpha - r\eta)z^3 + (\eta^2 - 4\alpha r\eta + \alpha^2 + 2\bar{\alpha}\bar{\eta})z^2 \\ + 2(\alpha\eta^2 - 2r\bar{\alpha} - r\alpha^2\eta + \bar{\eta} + |\alpha|^2\bar{\eta})z \\ + \alpha^2\eta^2 + \bar{\alpha}^2\bar{\eta}^2 + 2(\bar{\alpha}\eta + \alpha\bar{\eta}) - 4r(1 + |\alpha|^2) + \cdots + 1/z^4,$$

where $r = \frac{1}{2}(\rho + 1/\rho)$. Compare coefficients of (10) and (15),

$$b_0 = \alpha - r\eta,$$

$$b_0^2 - b_1 + 1 = \eta^2 - 4\alpha r\eta + \alpha^2 + 2\bar{\alpha}\bar{\eta}.$$

Eliminating α between these two equalities leads to (11) and (12).

Equation (9) becomes

$$z^2 f'(z)^2 [f(z)^2 + 1 - b_1] = \left(z + b_0 + r\eta + \frac{\bar{b}_0\bar{\eta} + r\bar{\eta}^2}{z} + \frac{\bar{\eta}}{z^2} \right)^2 (z^2 - 2r\eta z + \eta^2),$$

which can be rewritten as

$$(f^2 + 1 - b_1)^{\frac{1}{2}} df = \left(1 + \frac{b_0 + r\eta}{z} + \frac{\bar{b}_0\bar{\eta} + r\bar{\eta}^2}{z^2} + \frac{\bar{\eta}}{z^3} \right) (z^2 - 2r\eta z + \eta^2)^{\frac{1}{2}} dz.$$

We introduce the notation

$$Q = (z^2 - 2r\eta z + \eta^2)^{\frac{1}{2}}$$

and integrate the last differential equation to obtain

$$(16) \quad \begin{aligned} & \frac{1}{2}f(f^2 + 1 - b_1)^{\frac{1}{2}} + \frac{1 - b_1}{2} \log[f + (f^2 + 1 - b_1)^{\frac{1}{2}}] + K \\ &= \frac{1}{2}Q \left(z + r\eta + 2b_0 - \frac{r\bar{\eta}^2 + 2\bar{b}_0\bar{\eta}}{z} - \frac{\bar{\eta}}{z^2} \right) \\ &+ \frac{1 - b_1}{2} \log(Q + z - r\eta) - \frac{1 - \bar{b}_1}{2} \log \left(r - \frac{Q + \eta}{z} \right), \end{aligned}$$

where K is a constant of integration.

In order to evaluate K , we substitute first $\rho\eta$ and then η/ρ for z into (16), using the fact $f(\rho\eta) = f(\eta/\rho) = \pm(b_1 - 1)^{1/2} = u$, say. We find successively that

$$K + \frac{1 - b_1}{2} \log u = \frac{1 - b_1}{2} \log \frac{\eta}{2} \left(\rho - \frac{1}{\rho} \right) - \frac{1 - \bar{b}_1}{2} \log \frac{1}{2} \left(\rho - \frac{1}{\rho} \right)$$

and

$$K + \frac{1 - b_1}{2} \log u = \frac{1 - b_1}{2} \log \frac{-\eta}{2} \left(\rho - \frac{1}{\rho} \right) - \frac{1 - \bar{b}_1}{2} \log \frac{-1}{2} \left(\rho - \frac{1}{\rho} \right).$$

From these two equations we conclude that b_1 is real and so

$$(17) \quad K = \frac{1 - b_1}{2} \log \frac{\eta}{u} = \frac{1 - b_1}{4} \log \frac{-\eta^2}{1 - b_1}.$$

From (16) and (17) we obtain (14).

On the other hand, K can also be evaluated by expanding both sides of (16) about the point at infinity. This yields the relation

$$\begin{aligned} & \frac{f^2}{2} - \frac{1 - b_1}{4} + \frac{1 - b_1}{2} \log 2f + K + 0 \left(\frac{1}{f} \right) \\ &= \frac{z^2}{2} + b_0z + \frac{(1 - 3r^2)\eta^2}{4} - b_0r\eta - \bar{b}_0\bar{\eta} - \frac{r\bar{\eta}^2}{2} \\ &+ \frac{1 - b_1}{2} \log 2z - \frac{1 - b_1}{2} \log(r - 1) + 0 \left(\frac{1}{z} \right). \end{aligned}$$

Substituting for f the expansion (1) and letting $z \rightarrow \infty$ we have

$$(18) \quad K = \frac{1}{4}(1 - 3r^2)\eta^2 - b_0r\eta - \bar{b}_0\bar{\eta} - \frac{r\bar{\eta}^2}{2} - \frac{b_0^2}{2} - \frac{3b_1 + 1}{4} - \frac{1 - b_1}{2} \log(r - 1).$$

Now (13) follows from (17) and (18) and Lemma 3 is proved.

3. COEFFICIENT INEQUALITIES

By means of the above three lemmas, we can prove the following theorems.

Theorem 1. *Let the function (1) belong to Σ . Then*

$$\operatorname{Re}\{b_3 + 4b_2 + 5b_1\} \leq 5.$$

Equality holds only for the function (3).

Proof. Let $L(f) = b_3 + 4b_2 + 5b_1$, then L is a linear functional over Σ . Since L does not depend on the coefficient b_0 , we may set $b_0 = 2$ so that (4) is satisfied. Let $f(z) = z + 2 + \cdots$ be a support point of Σ associated with the functional L . The omitted set Γ by f satisfies the differential equation (5). It follows from Lemmas 1, 2, and 3 that if $f \neq z + 2 + 1/z$, then there exists $r > 1$ and $\eta = e^{i\theta}$, θ real, such that (11)–(13) are satisfied. If we can show that such r and η do not exist, then our theorem is proved.

Substituting 2 for b_0 into (11), (12), and (13), we have, respectively,

$$(19) \quad (1+r)(1-2r)\sin 2\theta - 4(1+r)\sin \theta = 0,$$

$$(20) \quad x = (1-r)(1+3r)\cos 2\theta + 4(1-r)\cos \theta,$$

where $x = 1 - b_1$, and

$$(21) \quad \frac{1-b_1}{2} \log \frac{-(r-1)^2\eta^2}{1-b_1} - 2(1-b_1) + 2r\eta + 6\bar{\eta} + 2r\bar{\eta}^2 + 6 = 0.$$

Equation (19) has three solutions for θ in $(-\pi, \pi]$: $\theta = 0, \pi$ and $\cos \theta = 2/(1-3r)$. If $\theta = 0$, (19) gives for $r > 1$,

$$x = 5 - 2r - 3r^2 < 0,$$

and if $\theta = \pi$, (19) becomes

$$x = -3 + 6r - 3r^2 = -3(r-1)^2 < 0.$$

Both contradict $x = 1 - b_1 > 0$. If

$$(22) \quad \cos \theta = 2/(1-3r),$$

equation (20) becomes

$$(23) \quad x = \frac{3(1-r)^2(5+6r+9r^2)}{(1-3r^2)}.$$

Taking real part on the left-hand side of (21), we have

$$\frac{x}{2} \log \frac{(r-1)^2}{x} - 2x + 6 + 2(r+3)\cos \theta + 2r\cos 2\theta = 0.$$

Combining this with (22) and (23) and simplifying the results, we get

$$(5+6r+9r^2)[2(1-r)^2 + \log 3(5+6r+9r^2)] + 6(r-1) = 0.$$

Clearly, the expression in the left-hand side is positive for $r > 1$ and hence this equation has no solution for $r > 1$. The proof is completed.

Theorem 2. *Let the function $f(z)$ belong to Σ . Then*

$$(24) \quad \operatorname{Re} \left\{ b_3 + b_2 + \frac{5}{4}b_1 \right\} \leq \frac{5}{4}.$$

Equality occurs if and only if $f(z)$ is the function given by (3).

Proof. The support point

$$f(z) = z + 1 + b_1/z + \dots$$

of Σ associated with the functional $L(f) = b_3 + b_2 + \frac{5}{4}b_1$ maps Δ onto the w -plane omitting a set Γ , which satisfies the differential equation (5). Applying Lemmas 1 and 2, we see that if $f \neq z + 1 + 1/z$, then Γ contains just one root of the equation $w^2 = b_1 - 1$. By Lemma 3, there exists $r > 1$ and $\eta = e^{i\theta}$, θ real, such that (11)–(13) are satisfied. Since $b_0 = 1$, these equations become

$$(25) \quad 2(1+r)[(1-2r)\cos\theta - 1]\sin\theta = 0,$$

$$(26) \quad 1 - b_1 = (1-r)(1+3r)\cos 2\theta - 2\cos\theta,$$

and

$$(27) \quad \frac{1-b_1}{2} \log \frac{(r-1)^2 \eta^2}{1-b_1} - 2(1-b_1) + 3 + r\eta + 3\bar{\eta} + 2r\bar{\eta}^2 = 0.$$

Equation (25) has three roots for θ in the interval $(-\pi, \pi]$.

If $\theta = 0$, (26) gives for $r > 1$,

$$1 - b_1 = (1-r)(3r-1) < 0,$$

which is impossible.

If $\theta = \pi$, (26) gives

$$1 - b_1 = 3(1-r)(1+r) < 0$$

for $r > 1$, which is again impossible.

If

$$(28) \quad \cos\theta = 1/(1-2r),$$

then (26) and (27) becomes, respectively,

$$1 - b_1 = (r-1)(2-12r-9r^2+27r^3)/(1-3r)^2$$

and

$$\frac{1-b_1}{2} \log \frac{(r-1)^2}{1-b_1} - 2(1-b_1) + \frac{3-8r+9r^2-18r^3}{(1-3r)^2} = 0.$$

Eliminate $1 - b_1$ between the last two equations to obtain

$$(29) \quad \frac{1}{2}(r-1)(2-12r-9r^2+27r^3) \log \frac{(r-1)(1-3r)^2}{2-12r-9r^2+27r^3} + 7-36r+15r^2+54r^3-54r^4 = 0.$$

But for $r > 1$ we have

$$(r-1)(2-12r-9r^2+27r^3) > 0, \\ (r-1)(1-3r)^2 < 2-12r-9r^2+27r^3,$$

and

$$7-36r+15r^2+54r^3-54r^4 < 0.$$

Hence the expression on the left-hand side of (29) is negative over $r > 1$, and so the equation (29) has no solution for $r > 1$. This completes the proof of Theorem 2.

Theorem 3. *Let the function (1) be in Σ . Then*

$$\operatorname{Re}\{b_3 + b_1\} \leq 1.$$

Equality holds only for the function (3).

Proof. Let $b_0 = 0$ and let

$$f(z) = z + b_1/z + \dots$$

be a support point of Σ associated with the linear functional $L(f) = b_3 + b_1$. Then the omitted arcs Γ by f satisfy the differential equation (5). It follows from Lemmas 1 and 2 that if $f(z) \neq z + 1/z$, then Γ contains one and only one root of the equation $w^2 = b_1 - 1$. By Lemma 3, there exists $r > 1$ and $\eta = e^{i\theta}$, with θ real, such that (11), (12), and (13) hold with $b_0 = 0$. That is,

$$(30) \quad (1+r)(1-3r)\sin 2\theta = 0,$$

$$(31) \quad 1 - b_1 = (1-r)(1+3r)\cos 2\theta,$$

and

$$(32) \quad \frac{1-b_1}{2} \log \frac{(r-1)^2}{1-b_1} - 2(1-b_1) + 2(1-r) = 0.$$

The condition (30) yields $\sin 2\theta = 0$ so that (31) gives

$$1 - b_1 = (r-1)(1+3r)$$

since $1 - b_1 > 0$. Substituting this value for $1 - b_1$ into (32), we obtain

$$(1-r) \frac{1+3r}{2} \log \frac{r-1}{1+3r} - 4 - 2r + 6r^2 = 0.$$

But this equation has no solution on $1 < r < +\infty$ as it is readily seen that the left-hand side is positive over $r > 1$. Hence we must have $f(z) = z + 1/z$ and Theorem 3 is proved.

The extremal functions in Theorems 1, 2, and 3 are the same function (3). Now we provide an extremal problem whose extremal function differs from (3), although the omitted set satisfies the differential equation (5).

Theorem 4. *Let the function (1) belong to Σ . Then*

$$\operatorname{Re}\{b_3 + 2ib_2\} \leq \frac{1}{2}r^3 - \frac{7}{2}r^2 + 4r + 1 = 1.138928\dots,$$

where r is the root of the equation

$$(33) \quad (3r^2 - 4r + 1) \log \frac{r-1}{3r-1} - 12r^2 + 14r + 4 = 0.$$

Equality holds only for the function given by the implicit equation:

$$(34) \quad \begin{aligned} & f(f^2 + c)^{\frac{1}{2}} + c \log[f + (f^2 + c)^{\frac{1}{2}}] \\ &= \left[z - i(r-2) + \frac{r-2}{z} - \frac{i}{z^2} \right] Q + c \log c \\ &+ \log(Q + z + ir) - \log \left(1 - \frac{Q-i}{z} \right), \end{aligned}$$

where $c = 3r^2 - 4r + 1$, $Q = (z^2 + 2irz - 1)^{1/2}$.

Proof. Let $b_0 = i$ and let

$$f(z) = z + i + b_1/z + \dots$$

be a support point of Σ associated with the linear functional $L(f) = b_3 + 2ib_2$. Then the omitted set Γ by f satisfies (5). Clearly $f \neq z + 1/z + i$. By Lemmas 1 and 2 the set Γ contains one and only one root of $w^2 = b_1 - 1$. By Lemma 3 we can find $r > 1$ and $\eta = e^{i\theta}$, θ real, such that (11)–(14) are satisfied. That is, we have

$$(35) \quad 2(r+1)[(1-3r)\sin\theta - 1]\cos\theta = 0,$$

$$(36) \quad 1 - b_1 = (r-1)[2\sin\theta - (1-3r)\cos\theta],$$

$$(37) \quad (1-b_1)\log\frac{-(r-1)^2\eta^2}{1-b_1} + 2 - 4(1-b_1) + 2ir\eta - 6i\bar{\eta} + 4r\bar{\eta}^2 = 0.$$

Equation (35) has three solutions: $\sin\theta = 1/(1-3r)$, $\theta = \pi/2$, $\theta = -\pi/2$.

If $\sin\theta = 1/(1-3r)$, (36) becomes

$$1 - b_1 = 3(1-r)(9r^3 - 3r^2 - r - 1)/(3r-1)^2,$$

which is negative for $r > 1$. This contradicts $1 - b_1 > 0$.

If $\theta = \pi/2$, (36) gives

$$1 - b_1 = 3r^2 + 1 > 4 \quad \text{for } r > 1,$$

which is impossible since $1 - b_1 < 2$.

The only solution is hence $\theta = -\pi/2$ or $\eta = -i$. Now (36) and (37) become, respectively,

$$(38) \quad 1 - b_1 = 3r^2 - 4r + 1$$

and

$$(1-b_1)\log\frac{(r-1)^2}{1-b_1} - 4(1-b_1) - 2r + 8 = 0.$$

Eliminating $1 - b_1$ between these two equations we obtain (33). For $r > 1$, (33) has only one root $r = 1.29774029\dots$. Thus r and $\eta = i$ are uniquely determined. With these values of r and η , letting $b_0 = i$ and using (38), we obtain from (14) the implicit equation for the extremal function $f(z)$.

To establish the inequality of the theorem, we have to represent $b_3 + 2ib_2$ in term of r . Letting $b_0 = i$ in (10) and $\alpha = i(1-r)$, $\eta = -i$ in (15), we deduce

$$(39) \quad \begin{aligned} z^4 + 2iz^3 - b_1z^2 - 2(b_2 + ib_1)z - 2(2b_3 + 3ib_2) + \dots + 1/z^4 \\ = z^4 + 2iz^3 - r(3r-4)z^2 - 2i(1+r^2-r^3)z \\ - 2(2r^3 - 5r^2 + 4r + 1) + \dots + 1/z^4. \end{aligned}$$

Comparing the coefficients of z on both sides,

$$b_2 + ib_1 = i(1 + r^2 - r^3).$$

Combining this with (38),

$$ib_2 = -r^3 - 2r^2 + 4r + 1.$$

Identify the constant terms on both sides of (39) to obtain

$$2b_3 + 3ib_2 = 2r^3 - 5r^2 + 4r + 1.$$

Thus we have

$$b_3 = \frac{5}{2}r^3 + \frac{1}{2}r^2 - 4r - 1, \quad b_3 + 2ib_2 = \frac{1}{2}r^3 - \frac{7}{2}r^2 + 4r + 1.$$

Numerically,

$$b_1 = -0.138572\dots, \quad ib_2 = 0.637140\dots, \quad b_3 = 0.115008\dots,$$

and

$$b_3 + 2ib_2 = 1.389288\dots$$

The proof is complete.

Remark. Since the centroid of Γ omitted by f given by (34) is $b_0 = i$, Γ must fork at the point $i(1 - b_1)^{1/2} = (1.067\dots)i$ in three directions. One arc of Γ is a segment lying on the imaginary axis with $i(1 - b_1)^{1/2}$ as the lower end.

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