ON THE MINIMUM MODULUS OF TRIGONOMETRIC POLYNOMIALS

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ABSTRACT. For all even integers N greater than 2, a trigonometric polynomial $f_N(x) = \sum_{k=-N}^{N^2} a_k e^{ikx}$ satisfying $|a_k| \leq 1$ and $0.47N \leq |f_N(x)| \leq N$ is constructed.

In this paper we are concerned with the problem of finding polynomials P_n of degree n whose coefficients are bounded by 1 and whose modulus $|P_n(z)|$ on $|z| = 1$ exceeds $A\|P_n\|_\infty$, for some absolute constant $A > 0$. This problem is part of the more general question, as expressed by Beller [Be]: "How close can we get to a situation where $P(z)$ is a polynomial of degree $n > 0$ which, on the one hand, has coefficients of constant modulus, and on the other hand $|P(z)|$ is constant for $|z| = 1$?" The question goes back to Littlewood [L] and has also been studied by Beller and Newman [BeN], Byrnes [By], Körner [Ko], and Kahane [Ka]. Problem 27 of Erdős [E] is also concerned with this topic. From the standpoint of existence, the question has been emphatically answered by Kahane [Ka] where he proves the following.

Theorem. *There exists a sequence of polynomials*

$$P_n(z) = \sum_{m=1}^n a_{m,n} z^m$$

with $|a_{m,n}| = 1$ and a positive sequence ε_n converging to 0 such that for all $|z| = 1$ the following holds:

$$(1 - \varepsilon_n)\sqrt{n} \leq |P_n(z)| \leq (1 + \varepsilon_n)\sqrt{n}.$$

However, Kahane's proof is probabilistic and it is not known how to construct such polynomials. It is not even known how to construct polynomials which satisfy

$$A\sqrt{n} \leq |P_n(z)| \leq B\sqrt{n}$$

for some pair of absolute constants $0 < A < B < \infty$.

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The upper bound inequality is not so hard to deal with. Beller [Be] gives polynomials P_n which satisfy

$$|P_n(z)| \leq 1.717\sqrt{n} \quad \text{for } |z| = 1$$

for n sufficiently large. It is even possible to make the coefficients ± 1 . The Rudin-Shapiro polynomials [R, S] satisfy

$$|P_n(z)| \leq \sqrt{2}\sqrt{n} \quad \text{for } |z| = 1$$

but are defined only for $n = 2^k$. They can be concatenated to give

$$|P_n(z)| \leq (\sqrt{2}/(\sqrt{2} - 1))\sqrt{n} \quad \text{for } |z| = 1$$

for arbitrary n .

Keeping $P_n(z)$ bounded away from 0 is a much more delicate matter. No example is known of polynomials $P_n(z) = \sum_{m=1}^n a_{m,n} z^m$ satisfying

- (1) $A\|P_n\|_\infty \leq |P_n(z)|$ for $|z| = 1$.
- (2) $|a_{m,n}| = 1$ for $n = 1, 2, \dots; m = 1, \dots, n$.

In [BeN] Beller and Newman give a construction of polynomials P_n with coefficients $|a_{m,n}| \leq 1$ which satisfy (1) with a constant A of about 0.0236. In this paper we give a construction based on a different idea which yields a constant A of about 0.47.

Theorem. *Let N be an even integer greater than 2. There exists a trigonometric polynomial*

$$P_N(x) = \sum_{k=-N}^{N^2} a_k e^{ikx}$$

satisfying

- (1) $|a_k| \leq 1$ for $k = -N, \dots, N^2$,
- (2) $0.47N \leq |P_N(x)| \leq N$ for all real x .

Proof. Let

$$K_N(x) = \frac{1}{N} \left(\frac{\sin Nx/2}{\sin x/2} \right)^2.$$

Then the Fourier transform is

$$\hat{K}_N(n) = \left(1 - \frac{|n|}{N} \right) \quad \text{for } 0 \leq |n| \leq N$$

and $\hat{K}_N(n) = 0$ otherwise. Let σ be the permutation of $(0, 1, \dots, N-1)$ defined by

$$\sigma = (0, 2, 4, \dots, N-2, N-1, N-3, \dots, 3, 1).$$

Note that σ^{-1} is given by

$$\sigma^{-1} = (0, N-1, 1, N-2, 2, N-3, \dots, N/2).$$

Let $\varepsilon(n)$ for $n = 0, \dots, N-1$ be defined by

$$\varepsilon(n) = \begin{cases} 1 & \text{for } n = 0, \dots, (N/2) - 1, \\ -1 & \text{for } (N/2), \dots, N-1. \end{cases}$$

Let

$$f_k(x) = \varepsilon(k)e^{i\sigma(k)Nx} K_N \left(x - \frac{2\pi k}{N} \right).$$

Let

$$P_N = \sum_{k=0}^{N-1} f_k.$$

We will first show that $\|\hat{P}_N\|_\infty \leq 1$. Note that

$$\hat{f}_k(n) = \varepsilon(k)e^{-2\pi i k n/N} \hat{K}_N(n - \sigma(k)N).$$

Denote by A_j the arithmetic progression $jN, jN+1, \dots, (j+1)N-1$. Then $\hat{f}_k$ is supported on the set $A_{\sigma(k)-1} \cup A_{\sigma(k)}$. This means that on any A_j at most two of the $\hat{f}_k$ are nonvanishing. In particular,

$$\hat{P}_N(n) = \begin{cases} \hat{f}_{\sigma^{-1}(0)}(n) & \text{for } n \in A_{-1}, \\ \hat{f}_{\sigma^{-1}(j)}(n) + \hat{f}_{\sigma^{-1}(j+1)}(n) & \text{for } n \in A_j, j = 0, \dots, N-2, \\ \hat{f}_{\sigma^{-1}(N-1)}(n) & \text{for } n \in A_{N-1}. \end{cases}$$

Since

$$|\hat{f}_{\sigma^{-1}(j)}(n)| = \left(1 - \frac{|n - jN|}{N} \right),$$

it follows that $|\hat{P}_N(n)| \leq 1$ for all n .

We next show that $\|P_N\|_\infty = N$. Let

$$S_N(x) = \sum_{k=0}^{N-1} K_N \left(x - \frac{2\pi k}{N} \right).$$

Then $|P_N(x)| \leq S_N(x) = K_N * \mu(x)$ where $\mu = \sum_{k=0}^{N-1} \delta_{2\pi k/N}$. Since $\text{supp } \hat{K}_N \subset \{-N, \dots, N\}$ and $\text{supp } \hat{\mu} = N\mathbb{Z}$ it follows that $\hat{S}_N = \hat{K}_N \hat{\mu} = \delta_0$ which shows that S_N is a constant. Evaluating $S_N(0) = K_N(0) = N$ gives $\|P_N\|_\infty \leq N$. But $P_N(0) = N$ so that $\|P_N\|_\infty = N$.

Finally, we estimate $\min |P_N(x)|$. Suppose $x \in I_{k_0}$ where I_k is the interval defined by $I_k = [2\pi k/N, 2\pi(k+1)/N)$. Let

$$\begin{aligned} R_{N, N-1} &= f_{N-1} + f_0, \\ R_{N, k_0} &= f_{k_0} + f_{k_0+1} \quad \text{for } 0 \leq k_0 < N-1, \\ Q_{N, k_0} &= P_N - R_{N, k_0} \quad \text{for } 0 \leq k_0 \leq N-1. \end{aligned}$$

For $x \in I_{k_0}$

$$|R_{N, k_0}(x)| - |Q_{N, k_0}(x)| \leq |P_N(x)| \leq |R_{N, k_0}(x)| + |Q_{N, k_0}(x)|.$$

We will estimate $|R_{N, k_0}(x)|$ and $|Q_{N, k_0}(x)|$ on I_{k_0} . Write

$$\begin{aligned} (1) \quad |R_{N, k_0}(x)| &= |f_{k_0} + f_{k_1}| \\ &= \left| K_N \left(x - \frac{2\pi k_0}{N} \right) + \eta e^{i\beta N x} K_N \left(x - \frac{2\pi k_1}{N} \right) \right| \end{aligned}$$

where $\eta = \varepsilon(k_1)/\varepsilon(k_0)$ and $\beta = \sigma(k_1) - \sigma(k_0)$. The definitions of σ and ε insure that β is either ± 1 or ± 2 and that

$$\eta = \begin{cases} +1 & \text{when } \beta = \pm 2, \\ -1 & \text{when } \beta = \pm 1. \end{cases}$$

Consider first the case $k_0 = N - 1$. Then $k_1 = 0$, $\eta = -1$, $\beta = -1$. Using (1) we have

(2)

$$|R_{N, N-1}(x)| = \left[K_N^2 \left(x + \frac{2\pi}{N} \right) - 2 \cos Nx K_N \left(x + \frac{2\pi}{N} \right) K_N(x) + K_N^2(x) \right]^{1/2}.$$

Also using the earlier result that $S_N(x) = N$ for all x gives

$$\begin{aligned} (3) \quad |Q_{N, N-1}(x)| &\leq \sum_{k=1}^{N-2} K_N \left(x - \frac{2\pi k}{N} \right) = S_N(x) - K_N(x) - K_N \left(x + \frac{2\pi}{N} \right) \\ &= N - K_N(x) - K_N \left(x + \frac{2\pi}{N} \right). \end{aligned}$$

Tedious but elementary calculations can be used to find the minimum of

$$\begin{aligned} &\left[K_N^2 \left(x + \frac{2\pi}{N} \right) - 2 \cos Nx K_N \left(x + \frac{2\pi}{N} \right) K_N(x) + K_N^2(x) \right]^{1/2} \\ &\quad - N + K_N(x) + K_N \left(x + \frac{2\pi}{N} \right) \end{aligned}$$

on the interval $[2\pi(N-1)/N, 2\pi]$. This minimum is found to exceed $0.619N$. Hence

$$0.619N \leq |P_N(x)| \leq N \quad \text{for } x \in I_{N-1}.$$

If $k_0 = (N/2) - 1$ then $\eta = -1$ and $\beta = 1$. A similar analysis as in the previous case gives the same bounds on $|P_N(x)|$ for $x \in I_{(N/2)-1}$.

Next consider $k_0 \in \{0, \dots, (N/2) - 2\} \cup \{(N/2), \dots, N - 2\}$. Then $\eta = 1$ and $\beta = \pm 2$. Using (1) we find

$$\begin{aligned} |R_{N, k_0}(x)| &= \left[K_N^2 \left(x - \frac{2\pi k_0}{N} \right) + K_N^2 \left(x - \frac{2\pi(k_0+1)}{N} \right) \right. \\ &\quad \left. + 2 \cos 2Nx K_N \left(x - \frac{2\pi k_0}{N} \right) K_N \left(x - \frac{2\pi(k_0+1)}{N} \right) \right]^{1/2}. \end{aligned}$$

As before we can write

$$|Q_{N, k_0}(x)| \leq N - K_N \left(x - \frac{2\pi k_0}{N} \right) - K_N \left(x - \frac{2\pi(k_0+1)}{N} \right).$$

Again, tedious but elementary calculations show that the minimum for $x \in I_{k_0}$ of

$$\begin{aligned} &\left[K_N^2 \left(x - \frac{2\pi k_0}{N} \right) + K_N^2 \left(x - \frac{2\pi(k_0+1)}{N} \right) \right. \\ &\quad \left. + 2 \cos 2Nx K_N \left(x - \frac{2\pi k_0}{N} \right) K_N \left(x - \frac{2\pi(k_0+1)}{N} \right) \right]^{1/2} \\ &\quad - N + K_N \left(x - \frac{2\pi k_0}{N} \right) + K_N \left(x - \frac{2\pi(k_0+1)}{N} \right) \end{aligned}$$

exceeds $0.47N$. Therefore

$$0.47N \leq |P_N(x)| \leq N \quad \text{for } x \in I_{k_0}. \quad \square$$

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