PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 114, Number 3, MARCH 1992

CLOSURE SETS OF FUNCTIONS AND A HIERARCHY OF FILTERS

ROBERT MIGNONE

(Communicated by Andreas R. Blass)

ABSTRACT. This paper presents a transfinite extension of a filter generating closure property of functions. One consequence of this extension is a hierarchy of filters which coincide with filters generated by a directed set type closure property. At each level of this hierarchy a progressively stronger notion of normality is satisfied. Ideals with this stronger notion of normality each have a corresponding saturation characterization.

Normality for filters on $P_\kappa\lambda$ can be defined in terms of regressive functions from $P_\kappa\lambda$ into λ. It is well known that an equivalent definition for normality can be given in terms of regressive functions from $P_\kappa\lambda$ into $P_\omega\lambda$. Likewise, the minimal normal filter can be characterized as the filter generated by sets closed under functions from $\lambda \times \lambda$ into $P_\kappa\lambda$ and an equivalent characterization can be made using functions from $P_\omega\lambda$ into $P_\kappa\lambda$. What if these two notions were extended to functions from $P_\kappa\lambda$ into $P_\delta\lambda$ and functions from $P_\delta\lambda$ into $P_\kappa\lambda$ respectively, for appropriate $\delta < \kappa$? This question leads to the concepts and results in the present paper.

Standard notation is used throughout. Let κ be a regular cardinal. If $\lambda \geq \kappa$ is a cardinal with $cf(\lambda) \geq \kappa$,

$$P_\kappa\lambda = \{ x \subset \lambda : |x| < \kappa \}.$$

Given $x \in P_\kappa\lambda$, let $\hat{x} = \{ y \in P_\kappa\lambda : x \subseteq y \}$. If F is a filter on $P_\kappa\lambda$, F^* is its dual ideal and vice versa, and

$$F^+ = \{ A \subseteq P_\kappa\lambda : A \notin F^* \}.$$

The *final section filter* is given by

$$FSF_{\kappa\lambda} = \{ A \subseteq P_\kappa\lambda : \hat{x} \subseteq A \text{ for some } x \in P_\kappa\lambda \}.$$

Filters over $P_\kappa\lambda$ extending $FSF_{\kappa\lambda}$ are called *fine*. And $A \subseteq P_\kappa\lambda$ is said to be *unbounded* if $A \cap \hat{x} \neq \varnothing$ for all $x \in P_\kappa\lambda$. For the extent of this paper, all filters

Received by the editors April 20, 1990 and, in revised form, September 14, 1990; the results contained in this paper were presented in University Park, Pennsylvania, April 1990, during the spring meeting of the American Mathematical Society.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 04A20.

This research was partially supported by a College of Charleston Research Grant and a Southeastern Regional Education Board Grant. The author also wishes to express his appreciation to MSRI, where this research was carried out.

(ideals) will be κ-complete extensions of $FSF_{\kappa\lambda}$ ($FSF_{\kappa\lambda}{}^*$). A set $C \subseteq P_\kappa\lambda$ is said to be *closed under chains,* if whenever $\{x_\gamma : \gamma < \delta\} \subset C$ where $x_\gamma \subseteq x_{\gamma'}$, when $\gamma < \gamma' < \delta < \kappa$, then $\bigcup\{x_\gamma : \gamma < \delta\} \in C$. The *closed unbounded filter* is given by

$$CF_{\kappa\lambda} = \{A \subseteq P_\kappa\lambda : C \subseteq A \text{ for some closed unbounded set } C\}.$$

Let

$$\bar{\delta} = \inf\{\delta \leq \kappa : \exists\gamma < \kappa \text{ and } \gamma^{<}\delta \geq \kappa\}.$$

Definition 1. For δ regular, let $v : P_\delta\lambda \longrightarrow P_\kappa\lambda$ and

$$C(v) = \{x \in P_\kappa\lambda : y \in \mathrm{P}_\delta x \Longrightarrow v(y) \subset x\}.$$

If $x \in C(v)$, then x is said to be *closed under* v and $C(v)$ is called the *closure set of* v.

Definition 2. Let δ be regular and $\delta < \bar{\delta}$. Given $v : P_\delta\lambda \longrightarrow P_\kappa\lambda$, define v' as follows: for $x \in P_\kappa\lambda$, let

$$v_1(x) = \bigcup\{v(y) : y \in P_\delta x\} \cup x;$$

and for $1 < \gamma < \delta$,

$$v_\gamma(x) = \bigcup\left\{v(y) : y \in P_\delta\left(\bigcup\{v_\xi(x) : \xi < \gamma\}\right)\right\}.$$

Set

$$v'(x) = \bigcup_{\gamma<\delta} v_\gamma(x).$$

Definition 3. For a regular $\delta < \kappa$, $D \subseteq P_\kappa\lambda$ is said to be *δ-directed,* if whenever $d \subset D$ and $|d| < \delta$, there exists a $y \in D$ such that $\bigcup d \subseteq y$. And $C \subseteq P_\kappa\lambda$ is said to be *δ-dclosed,* if whenever $D \subset C$ such that $|D| < \kappa$ and D is δ-directed, then $\bigcup D \in C$.

Theorem 4. *Let $\delta < \bar{\delta}$ be regular. Suppose that $v : P_\delta\lambda \longrightarrow P_\kappa\lambda$ and $w : P_\delta\lambda \longrightarrow P_\kappa\lambda$. Then*

(1) $C(v) = C(v')$;
(2) $\forall x \in P_\kappa\lambda(x \subseteq v'(x) \wedge v'(x) \in C(v'))$;
(3) $\forall x \in P_\delta\lambda(v'(x) = \bigcup\{v'(y) : y \in P_\delta(v'(x))\})$;
(4) $\forall x, y \in P_\delta\lambda((v'(x) \cap v'(y)) \in C(v'))$;
(5) $\forall x \in P_\delta\lambda(C(w) \subset C(v) \leftrightarrow v'(x) \subset w'(x))$;
(6) $\forall x \in P_\delta\lambda(v'(x)$ *is the smallest set closed under* v, *containing* $x)$;
(7) $C(v') = \{\bigcup v'[x] : x \in P_\kappa\lambda\}$, *where* $v'[x] = \{v'(y) : y \in P_\delta x\}$;
(8) $v'[x]$ *is δ-directed.*

Proof.

(1) $C(v') \subset C(v)$ since $\forall x \in P_\delta\lambda(v(x) \subset v'(x))$. Next, given $x \in C(v)$, let $y \in P_\delta x$. Then by induction on $\gamma < \delta$,

$$v_1(y) = \bigcup\{v(z) : z \in P_\delta y\} \cup y \subset x;$$

and

$$v_\gamma(y) = \bigcup\left\{v(z) : z \in P_\delta\left(\bigcup\{v_{\gamma'}(y) : \gamma' < \gamma\}\right)\right\} \subset x,$$

since $\forall \gamma' < \gamma (v_{\gamma'}(y) \subset x)$, by the induction hypothesis. Hence, $v'(y) \subset x$ and $x \in C(v')$.

(2) Clearly, $x \in P_\kappa(v'(x))$. And if $y \in P_\delta(v'(x))$, then $|y| < \delta$, so $y \subset v_\gamma(x)$ for some $\gamma < \delta$. Hence $v(y) \in v_{\gamma+1}(x)$.

(3) Let $\alpha \in v'(x)$. Then $\alpha \in v'(\{\alpha\}) \subset \bigcup\{ v'(y) : y \in P_\delta(v'(x))\}$. And if $\alpha \in \bigcup\{ v'(y) : y \in P_\delta(v'(x))\}$, then $\alpha \in v'(y) \subset v'(x)$ by (2).

(4) This holds since $v'(x)$ and $v'(y)$ are in $C(v)$ by (2).

(5) For the first direction, $x \in P_\delta(w'(x))$ and $w'(x) \in C(v) = C(v')$ by (1). Hence, $v'(x) \subset w'(x)$. For the remaining direction, if $x \in C(w)$ and $y \in P_\delta x$, then $v'(y) \subset w'(y) \subset x$. Hence, $x \in C(v)$.

(6) Given $x \in P_\delta\lambda$, let $z \in C(v)$ such that $x \subset z$. Then $v'(x) \subset z$.

(7) Let $x \in C(v')$. Then $x = \bigcup v'[x]$. This is so, since given $\alpha \in x$, then $\alpha \in v'(\{\alpha\}) \in \{ v'(y) : y \in P_\delta x\}$. And if $\alpha \in \bigcup v'[x]$, then $\alpha \in v'(y)$ and $y \in P_\delta x$, so $\alpha \in x$. Next, let $y \in P_\delta(\bigcup v'[x])$. Then $y \subset \bigcup\{ v'(z) : z \in P_\delta x\}$. Let $z_\alpha \in P_\delta x$ such that $\alpha \in v'(z_\alpha)$ for $\alpha \in y$. So $y \subset \bigcup\{ v'(z_\alpha) : \alpha \in y\} \subset v'(\bigcup\{ z_\alpha : \alpha \in y\})$. But $v'(\bigcup\{ z_\alpha : \alpha \in y\} \in C(v')$; hence $v'(y) \subset \bigcup v'[x]$.

(8) Let $d = \{ v'(y_\gamma) : \gamma < \xi\} \subset v'[x]$, where $y_\gamma \in P_\delta x$ and $\xi < \delta$. Since δ is regular, $|\bigcup\{ y_\gamma : \gamma < \xi\}| < \delta$, and hence $\bigcup\{ y_\gamma : \gamma < \xi\} \in P_\delta x$. By (3), $\bigcup d \subset v'(\bigcup\{ y_\gamma : \gamma < \xi\})$. □

The next definition strengthens the notion of normality for filters on $P_\kappa\lambda$.

Definition 5. A filter F on $P_\kappa\lambda$ is *δ-normal*, if whenever $f : P_\kappa\lambda \longrightarrow P_\delta\lambda$ such that $A \in F^+$ and $\forall x \in A(f(x) \in P_\delta x)$, then $\exists B \subseteq A$, and $y \in P_\delta\lambda$ such that $B \in F^+$ and $f(x) = y\ \forall x \in B$.

As in the case for normality, δ-normality can be characterized by a type of diagonal intersection. The symbol $\triangle^\delta$ will be used to indicate the diagonal intersection of a collection of subsets of $P_\kappa\lambda$ indexed over $P_\delta\lambda$.

Theorem 6. *A filter F on $P_\kappa\lambda$ is δ-normal if and only if whenever $\{ C_y : y \in P_\delta\lambda\} \subset F$ then*

$$\triangle^\delta\{ C_y : y \in P_\delta\lambda\} = \{ x \in P_\kappa\lambda : y \in P_\delta x \Longrightarrow x \in C_y\} \in F.$$

Proof. The usual argument adapts to this situation, see [Je1]. □

In [Me], Menas showed that the filter generated by closure sets of functions $v : P_\omega\lambda \longrightarrow P_\kappa\lambda$ is the same as the closed unbounded filter, $CF_{\kappa\lambda}$, on $P_\kappa\lambda$. In the same paper, Menas mentions and uses the fact that the filter generated by closure sets of functions $v : P_\omega\lambda \longrightarrow P_\kappa\lambda$ is the same as the filter generated by closure sets of functions $v : \lambda \times \lambda \longrightarrow P_\kappa\lambda$. This fact has proved useful in obtaining results involving the closed unbounded filter on $P_\kappa\lambda$. In particular, Carr used it to prove that $CF_{\kappa\lambda}$ is the minimal normal filter on $P_\kappa\lambda$, see [Ca]; Johnson used it to get results on M-ideals on $P_\kappa\lambda$, see [Jo]. Also in [Me], is the slightly hidden fact that the filter generated by the closure sets of functions $v : \lambda \longrightarrow P_\kappa\lambda$ is the same as $SCF_{\kappa\lambda}$, the strongly closed unbounded filter (which is the filter generated by unbounded subsets of $P_\kappa\lambda$ closed under arbitrary unions of size less than κ). In [Mi], this fact and a theorem similiar to Theorem 4 for $v : \lambda \longrightarrow P_\kappa\lambda$ are used to establish results about quasinormal filters on $P_\kappa\lambda$.

In what follows, a general theory is developed for filters generated by $v : P_\delta\lambda \longrightarrow P_\kappa\lambda$, where $\delta < \bar{\delta}$ is regular. What emerges is a strict linear hierarchy

of filters (ideals), each coinciding with a corresponding δ-closed unbounded filter and each satisfying a progressively stronger normality property, namely δ-normality.

Definition 7. Let $\delta < \bar{\delta}$ be regular. The *closure set filter* is given by

$$F^{\delta}_{\kappa\lambda} = \{ A \subset P_\kappa\lambda : \exists v : P_\delta\lambda \longrightarrow P_\kappa\lambda \text{ and } C(v) \subseteq A \}.$$

Theorem 8. *$F^{\delta}_{\kappa\lambda}$ is a δ-normal, κ-complete fine filter on $P_\kappa\lambda$.*

Proof. The verification that $F^{\delta}_{\kappa\lambda}$ is a κ-complete, fine filter is routine. Let

$$\{ C(v_y) : y \in P_\delta\lambda \} \subset F^{\delta}_{\kappa\lambda}.$$

Define $v(z) = \bigcup\{ v'_y(z) : y \in P_\delta z \}$. Let $x \in C(v)$, $y \in P_\delta x$ and $z \in P_\delta x$. Now

$$v(z \cup y) = \bigcup\{ v'_t(z \cup y) : t \in P_\delta(z \cup y) \}.$$

This gives $v_y(z) \subset v'_y(z \cup y) \subset v(z \cup y) \subset x$. The first inclusion follows from the construction of v'_y from v_y, the second inclusion from the definition of v and the third from the fact that $x \in C(v)$. $\square$

Definition 9. Let $\delta < \bar{\delta}$, be regular.

$$DCF^{\delta}_{\kappa\lambda} = \{ A \subset P_\kappa\lambda : \exists C \subseteq A \text{ where } C \text{ is } \delta\text{-dclosed unbounded} \}.$$

Note. If $C \subseteq P_\kappa\lambda$ is δ-dclosed unbounded, then C is closed under unions of chains from C of cofinal length greater than or equal to δ (call this *δ-closed*). The converse does not appear to yield to Solovay's method, see [Mag]. It would be interesting to determine whether or not a set is δ-dclosed unbounded if and only if it is δ-dclosed unbounded. Or, if the filter generated by δ-dclosed unbounded sets is denoted by $CF^{\delta}_{\kappa\lambda}$, then $DCF^{\delta}_{\kappa\lambda} \subset CF^{\delta}_{\kappa\lambda}$, but does equality hold?

Both $DCF^{\delta}_{\kappa\lambda}$ and $CF^{\delta}_{\kappa\lambda}$ are κ-complete, fine, δ-normal filters on $P_\kappa\lambda$. For $DCF^{\delta}_{\kappa\lambda}$ it is a consequence of the next theorem and for $CF^{\delta}_{\kappa\lambda}$ the verification is routine and will be omitted.

Theorem 10. *For $\delta < \bar{\delta}$, regular, $F^{\delta}_{\kappa\lambda} = DCF^{\delta}_{\kappa\lambda}$.*

Proof. Given $v : P_\delta\lambda \longrightarrow P_\kappa\lambda$, let $D \subset C(v)$ be δ-directed, where $|D| < \kappa$. Let $x \in P_\delta(\bigcup D)$. For each $\alpha \in x$, there exists a $x_\alpha \in D$ such that $\alpha \in x_\alpha$. There exists a $y \in D$ such that $\bigcup_{\alpha \in x} x_\alpha \subset y$. So $x \in P_\delta y$. This gives $v(x) \subset y \subset \bigcup D$. Note, $C(v)$ is unbounded by parts (1) and (2) of Theorem 4. Hence $F^{\delta}_{\kappa\lambda} \subset DCF^{\delta}_{\kappa\lambda}$.

Next, assume $A \subset P_\kappa\lambda$ is closed under δ-directed subsets of cardinality less than κ, and unbounded. Define $v_\gamma : P_\delta\lambda \longrightarrow P_\kappa\lambda$, for $\gamma < \delta$, as follows: for each $y \in P_\delta\lambda$,

$$v_1(y) \in A, \quad \text{where } y \subset v_1(y);$$
$$v_\gamma(y) \in A, \quad \text{where } \bigcup\{ v_{\gamma'}(z) : \gamma' < \gamma \text{ and } z \subset y \} \subset v_\gamma(y).$$

Claim. For every x in $P_\kappa\lambda$, $D_x = \{ v_\gamma(y) : \gamma < \delta \text{ and } y \in P_\delta x \}$ is a δ-directed subset of $P_\kappa\lambda$, of cardinality less than κ.

Proof of claim. Since $\delta < \bar{\delta}$, $|D_x| < \kappa$. So let

$$\{ v_{\gamma_\alpha}(y_\alpha) : \alpha < \delta' \} \subset D_x, \quad \text{where } \delta' < \delta.$$

Setting $\gamma = \sup\{ \gamma_\alpha : \alpha < \delta' \} + 1$ and $y = \bigcup_{\alpha<\delta'} y_\alpha$, gives $\gamma < \delta$ and $y \in P_\delta x$. By construction

$$\bigcup\{ v_{\gamma_\alpha}(y_\alpha) : \alpha < \delta' \} \subset v_\gamma(y) \in D_x.$$

This proves the claim.

Define $v : P_\delta\lambda \longrightarrow P_\kappa\lambda$ by $v(y) = \bigcup D_y$. Let $x \in C(v)$. The construction of v_γ and D_x yields $x \subset \bigcup D_x$. And if $\alpha \in \bigcup D_x$, then $\alpha \in v_\gamma(y)$ for some $y \in P_\delta x$. But

$$v_\gamma(y) \subset \bigcup D_y = v(y) \subset x.$$

Hence, $\bigcup D_x \subset x$. Since A is closed under δ-directed sets of cardinality less than κ, by the claim, $\bigcup D_x \in A$. Hence $x \in A$ and $C(v) \subset A$. This shows $DCF^\delta_{\kappa\lambda} \subset F^\delta_{\kappa\lambda}$. □

The next theorem shows that the $F^\delta_{\kappa\lambda}$ form a strictly increasing hierarchy.

Theorem 11. *$F^\delta_{\kappa\lambda} \subsetneq F^{\delta^+}_{\kappa\lambda}$, for all regular $\delta < \bar{\delta}$.*

Proof. Define $v : P_{\delta^+}\lambda \longrightarrow P_\kappa\lambda$ as follows:

$$v(x) = \{ \sup x \}.$$

The result will follow from Theorem 10 once it is demonstrated that no unbounded subset of $C(v)$ can be closed under δ-directed sets. Let $A \subset C(v)$ be an unbounded subset. Construct $\{ \alpha_\gamma : \gamma < \delta \}$ and $\{ x_\gamma : \gamma < \delta \}$ as follows: Let $\alpha_1 \in \lambda$, choose $x_1 \in A$ such that $\alpha_1 \in x_1$. If $\xi < \gamma$ and we have $\alpha_\xi \in \lambda$ and $x_\xi \in A$ defined, choose

$$\alpha_\gamma > \sup\left\{ \bigcup x_\xi : \xi < \gamma \right\}.$$

Now choose $x_\gamma \in A$ such that $\alpha_\gamma \in x_\gamma$ and $x_\xi \subset x_\gamma$ when $\xi < \gamma$. This shows that

$$\{ x_\gamma : \gamma < \delta \} \quad \text{is } \delta\text{-directed},$$
$$|\{ x_\gamma : \gamma < \delta \}| < \kappa,$$

and

$$\{ x_\gamma : \gamma < \delta \} \subset A.$$

Suppose $\bigcup\{ x_\gamma : \gamma < \delta \} \in A$. Consider $x = \{ \alpha_\gamma : \gamma < \delta \}$. Then

$$x \in P_{\delta^+}\left(\bigcup\{ x_\gamma : \gamma < \delta \} \right)$$

and

$$\bigcup\{ x_\gamma : \gamma < \delta \} \in C(v)$$

since $A \subset C(v)$. So

$$v(x) = \{ \sup x \} \subset \bigcup\{ x_\gamma : \gamma < \delta \}.$$

But clearly $\sup x \notin \bigcup\{ x_\gamma : \gamma < \delta \}$, a contradiction. □

The next set of definitions and theorem will be used to establish that, at each level of the hierarchy, $F^\delta_{\kappa\lambda}$ is the minimal δ-normal filter on $P_\kappa\lambda$.

Definition 12. For any filter F on $P_\kappa\lambda$, denote

$$\triangle F = \{ A \subset P_\kappa\lambda : \exists\{ A_\alpha : \alpha < \lambda \} \subset F \text{ and } \triangle_{\alpha<\lambda} A_\alpha \subseteq A \}.$$

In [Ca], Carr determined that for every filter F on $P_\kappa\lambda$ extending $FSF_{\kappa\lambda}$:

(1) $F \subseteq \triangle F$, where equality holds if F is normal;
(2) $\triangle FSF_{\kappa\lambda} = SCF_{\kappa\lambda} = (F^2_{\kappa\lambda})$;
(3) $\triangle\triangle FSF_{\kappa\lambda} = \triangle SCF_{\kappa\lambda} = CF_{\kappa\lambda}(= F^3_{\kappa\lambda} = F^\omega_{\kappa\lambda})$.

Definition 13. For any filter F on $P_\kappa\lambda$, let

$$\triangle^\delta F = \{ A \subseteq P_\kappa\lambda : \exists\{ A_y : y \in P_\delta\lambda \} \subset F \text{ and } \triangle^\delta \{ A_y : y \in P_\delta\lambda \} \subset A \}.$$

Theorem 14. *For regular* $\delta < \bar{\delta}$,

(1) $\triangle^\delta FSF_{\kappa\lambda} = F^\delta_{\kappa\lambda}$;
(2) $\triangle^\delta F^\xi_{\kappa\lambda} = F^\delta_{\kappa\lambda}$ *for all regular* $\xi < \delta$.

Proof. For the proof of part one, let $C \in \triangle^\delta FSF_{\kappa\lambda}$. Without loss of generality, assume

$$C = \triangle^\delta \{ \hat{x}_y : y \in P_\delta\lambda \},$$

where $x_y \in P_\kappa\lambda$ for $y \in P_\delta\lambda$. Define $v_y : P_\delta\lambda \longrightarrow P_\kappa\lambda$ by $v_y(z) = x_y$. Given

$$x \in \triangle^\delta \{ C(v_y) : y \in P_\delta\lambda \} \in F^\delta_{\kappa\lambda}.$$

If $y \in P_\delta x$, then $x \in C(v_y)$. So $x_y = v_y(z) \subset x$ for every $z \in P_\delta x$. Hence $x \in \hat{x}_y$, giving $x \in C$ and therefore $C \in F^\delta_{\kappa\lambda}$.

Next, let $v : P_\delta\lambda \longrightarrow P_\kappa\lambda$ and consider $C(v)$. For each $y \in P_\delta\lambda$, let $x_y \in C(v)$ such that $y \subset x_y$. Given $x \in \triangle^\delta \{ \hat{x}_y : y \in P_\delta\lambda \}$, let $y \in P_\delta x$. Then $x \in \hat{x}_y$ so $y \subset x_y \subset x$. Hence $v(y) \subset x_y \subset x$. Therefore, $x \in C(v)$ and $C(v) \in \triangle^\delta FSF_{\kappa\lambda}$.

For the proof of part two, let $C \in \triangle^\delta F^\xi_{\kappa\lambda}$, for any regular $\xi < \delta$. Without loss of generality, assume

$$C = \triangle^\delta \{ C(v_y) : y \in P_\delta\lambda \text{ and } v_y : P_\xi\lambda \longrightarrow P_\kappa\lambda \}.$$

Define $\bar{v}_y : P_\delta\lambda \longrightarrow P_\kappa\lambda$ by $\bar{v}_y(s) = \bigcup\{ v_y(t) : t \in P_\xi s \}$. Now

$$\triangle^\delta \{ C(\bar{v}_y) : y \in P_\delta\lambda \} \in F^\delta_{\kappa\lambda},$$

since $F^\delta_{\kappa\lambda}$ is δ-normal. Given $x \in \triangle^\delta \{ C(\bar{v}_y) : y \in P_\delta\lambda \}$, $y \in P_\delta x$ and $t \in P_\xi x$, then for any $s \in P_\delta x$ such that $t \subset s$, yields $v_y(t) \subset \bar{v}_y(s) \subset x$. Hence $x \in C$. Therefore, $C \in F^\delta_{\kappa\lambda}$.

Finally, let $v : P_\delta\lambda \longrightarrow P_\kappa\lambda$. For $y \in P_\delta\lambda$, define $v_y : P_\xi\lambda \longrightarrow P_\kappa\lambda$ by $v_y(t) \in C(v)$ such that $y \subset v_y(t)$ for every $t \in P_\xi\lambda$. This gives $C(v_y) \in F^\xi_{\kappa\lambda}$ for each $y \in P_\delta\lambda$. Let $x \in \triangle^\delta \{ C(v_y) : y \in P_\delta\lambda \}$. If $y \in P_\delta x$, then $x \in C(v_y)$. For any $t \in P_\xi x$, it is the case that $y \subset v_y(t)$ and $v_y(t) \in C(v)$. This gives $v(y) \subset v_y(t)$. But $x \in C(v_y)$, and hence $v(y) \subset v_y(t) \subset x$. Therefore $x \in C(v)$ and $C(v) \in \triangle^\delta F^\xi_{\kappa\lambda}$. □

Corollary 15. $F^\delta_{\kappa\lambda}$ *is the minimal* δ*-normal, fine,* κ*-complete filter on* $P_\kappa\lambda$.

Proof. Let F be any δ-normal, fine, κ-complete filter on $P_\kappa\lambda$. Then $FSF_{\kappa\lambda} \subset F$ and $\triangle^\delta FSF_{\kappa\lambda} \subset \triangle^\delta F = F$, giving $F^\delta_{\kappa\lambda} \subset F$. □

This section looks at the notions of stationarity and saturation related to the hierarchy of filters, $F^{\delta}_{\kappa\lambda}$ for $\delta < \bar{\delta}$ regular and presents some well-known results in this context.

Assume κ is inaccessible for the remainder of this section. The filters, $DCF^{\delta}_{\kappa\lambda}$ for $\delta < \bar{\delta}$ and their dual ideals $(DCF^{\delta}_{\kappa\lambda})^*$, are natural extensions of $CF_{\kappa\lambda}$ and the nonstationary ideal, $NS_{\kappa\lambda}$. Since $F^{\delta}_{\kappa\lambda} = DCF^{\delta}_{\kappa\lambda}$ it holds that $(F^{\delta}_{\kappa\lambda})^+ \subset CF^+_{\kappa\lambda}$.

Definition 16. Say $S \subset P_\kappa\lambda$ is *δ-stationary* if $A \cap S \neq \varnothing$, for all A in $F^{\delta}_{\kappa\lambda}$. That is, S is δ-stationary if and only if $S \in (F^{\delta}_{\kappa\lambda})^+$.

The following theorem is in the spirit of [Me, Corollary 16], and characterizes δ-stationary sets in terms of "δ-regressive" functions.

Theorem 17. *For $\delta < \bar{\delta}$, $S \subset P_\kappa\lambda$ is δ-stationary if and only if for every $f : P_\kappa\lambda \longrightarrow P_\delta\lambda$ such that $f(x) \in P_\delta x$, $\forall x \in S$ (say f is δ-regressive over S), there exists an unbounded subset of S where f is constant.*

Proof. One direction follows directly from the fact that $F^{\delta}_{\kappa\lambda}$ is δ-normal.

For the other direction, assume that S is not δ-stationary. Let $v : P_\delta\lambda \longrightarrow P_\kappa\lambda$ be such that $C(v) \subset P_\kappa\lambda - S$. Define $f : S \longrightarrow P_\delta\lambda$ by $f(x) \in P_\delta x$ such that $v(f(x)) \nsubseteq x$. But this leads to a contradiction, since $f^{-1}(\{y\}) \cap S$ cannot be unbounded for any $y \in P_\delta\lambda$. □

Recall that an ideal I is said to be *μ-saturated* if whenever $\{A_\gamma : \gamma < \nu\} \subset I^+$ such that $A_\gamma \cap A_{\gamma'} \in I$ for $\gamma \neq \gamma'$, then $\nu < \mu$.

The next theorem gives a characterization of saturation for δ-normal ideals on $P_\kappa\lambda$ in terms of their possible extensions by δ-normal ideals on $P_\kappa\lambda$. This type of characterization was first used in [B-T-W] for normal ideals on a cardinal κ.

Given an ideal I on $P_\kappa\lambda$, let

$$I|A = \{B \subset P_\kappa\lambda : A \cap B \in I\}.$$

This is the ideal generated by I and $P_\kappa\lambda - A$. Observe that if $A \in I^*$, then $I|A = I$ and if $A \in I$ then $I|A = P(P_\kappa\lambda)$. Hence $I|A$ is a proper extension of I when $A \in I^+ - I^*$.

Theorem 18. *Let $\delta < \kappa$. A δ-normal, fine ideal I on $P_\kappa\lambda$ is $(\lambda^{<\delta})^+$-saturated if and only if $I|A$ for $A \in I^+ - I^*$ are the only δ-normal ideals on $P_\kappa\lambda$ which properly extend I.*

Proof. The proof follows the argument of [B-T-W] with straightforward modifications. □

Mahlo's operation: for $A \subset \kappa$,

$$M(A) = \{\alpha < \kappa : cf(\alpha) > \omega \text{ and } A \cap \alpha \notin NS_\alpha\}$$

led to the notion of an M-ideal, which plays a major role in gaining results on splitting stationary sets on κ (see [B-T-W]), and on $P_\kappa\lambda$ (see [Jo]). In the latter case, Johnson first generalized the Mahlo operation to $P_\kappa\lambda$ as follows: for $A \subset P_\kappa\lambda$

$$M(A) = \{x \in A : x \cap \kappa \text{ is a weak. inacc. and } A \cap P_{x\cap\kappa}x \in NS^+_{x\cap\kappa,x}\}.$$

An *M-ideal* is an ideal I, which satisfies: $M(A) \in I^*$ if and only if $A \in I^*$.

Theorem 18, in combination with the following statement provide the base for the results on splitting stationary subsets in the two papers mentioned in the preceding paragraph: *If I is an M-ideal then, for any stationary subset A, $I \neq NS|A$ (where NS stands for the nonstationary ideal on κ or $P_\kappa\lambda$).* In the latter case, the fact that the closed unbounded filter on $P_\kappa\lambda$ can be replaced by the filter generated by closure sets of functions $v : P_3\lambda \longrightarrow P_\kappa\lambda$ provides the key to the proof of this statement.

A variation of Johnson's generalization of the Mahlo operation to $P_\kappa\lambda$ is presented, which is directly analogous to the way Jech generalized the diamond principle to $P_\kappa\lambda$ in [Je2].

Definition 19. For $A \subset P_\kappa\lambda$ and $\delta < \bar{\delta}$,

$$M^\delta(A) = \{ x \in A : |x| \text{ is weak. inacc., } |x| > \delta \text{ and } A \cap P_{|x|}x \in (F^\delta_{|x|,x})^+ \}.$$

And for I a δ-normal ideal, say I is an *M^δ-ideal* if $M^\delta(A) \in I^*$ whenever $A \in I^*$.

The usefulness of the filter generated by closure sets of functions $v : P_\delta\lambda \longrightarrow P_\kappa\lambda$ will be used to extend the statement mentioned in the preceding paragraph to this context.

Theorem 20. *If I is an M^δ-ideal on $P_\kappa\lambda$ then, for any $A \in (F^\delta_{\kappa\lambda})^+$, $I \neq (F^\delta_{\kappa\lambda})^*|A$.*

Proof. Suppose I is an M^δ-ideal and $I = (F^\delta_{\kappa\lambda})^*|A$ for some $A \in (F^\delta_{\kappa\lambda})^+$. Then there exists a $v : P_\delta\lambda \longrightarrow P_\kappa\lambda$ such that $C(v) \cap A \subseteq M^\delta(A)$.

Claim. There exists a $B \subset C(v)$ such that

$$B = \{ x \in P_\kappa\lambda : \forall y \in P_\delta x(\ v(y) \subset x \wedge |v(y)| < |x|) \}$$

and $B \in DCF^\delta_{\kappa\lambda}$.

Proof of claim. The proof of this claim is a straightforward exercise and will be omitted.

Let $x \in B \cap A \subset M^\delta(A)$ be such that $|x|$ is minimal. Define $v_x : P_\delta x \longrightarrow P_{|x|}x$ by $v_x(y) = v(y)$. This holds since $x \in C(v)$ and $y \in P_\delta x$, and hence $v_x(y) \subset x$. Also, $x \in B$ gives $|v_x(y)| < |x|$ and $v_x(y) \in P_{|x|}x$. But

$$C(v_x) \subset C(v) \cap P_{|x|}x,$$

since first: v_x assumes values in $P_{|x|}x$ so $C(v_x) \subset P_{|x|}x$; and if $y \in C(v_x)$ and $z \in P_\delta y \subset P_\delta x$, then $v(z) = v_x(z) \subset y$; hence, $y \in C(v)$. Next, let $B_x \subset C(v_x)$ such that

$$B_x = \{ y \in P_{|x|}x : \forall z \in P_\delta y(\ v_x(z) \subset y \wedge |v_x(z)| < |y|) \}.$$

Now $B_x \in F^\delta_{|x|,x}$ and hence $B \cap P_{|x|}x \in F^\delta_{|x|,x}$. But $x \in M^\delta(A)$, so $A \cap P_{|x|}x \in (F^\delta_{|x|,x})^+$. But this means that there exists $y \in B \cap A \cap P_{|x|}x$, contradicting the choice of x. $\square$

The theorems of this section are not meant to be exhaustive, but rather they are meant to suggest that a great deal of what can be done for $C_{\kappa\lambda}$ can in turn be done for $F^\delta_{\kappa\lambda}$. Another example of this point comes from analyzing the work of Matsubara in [Mat]. A consequence of Matsubara's technique is that $P_\kappa\lambda$ splits into $\lambda^{<\kappa}$ many disjoint δ-stationary subsets.

REFERENCES

[B-T-W] J. E. Baumgartner, A. D. Taylor, and S. Wagon, *On splitting stationary subsets of large cardinals*, J. Symbolic Logic **42** (1977), 203–214.

[Ca] D. M. Carr, *The minimal normal filter on* $P_\kappa\lambda$, Proc. Amer. Math. Soc. **86** (1982), 316–320.

[Je1] T. J. Jech, *Set theory*, Academic Press, 1978.

[Je2] ——, *Some combinatorial problems concerning uncountable cardinals*, Ann. of Math. Logic **5** (1973), 165–198.

[Jo] C. A. Johnson, *On saturated ideals and* $P_\kappa\lambda$, Fund. Math. **129** (1988), 215–221.

[Mag] M. Magidor, *Combinatorial characterization of supercompact cardinals*, Proc. Amer. Math. Soc. **42** (1974), 279–285.

[Mat] Y. Matsubara, *Splitting* $P_\kappa\lambda$ *into stationary subsets*, J. Symbolic Logic **53** (1988), 385–389.

[Me] T. K. Menas, *On strong compactness and supercompactness*, Ann. of Math. Logic **7** (1974), 327–359.

[Mi] R. J. Mignone, *A direct weakening of normality for filters* (to appear).

MATHEMATICAL SCIENCES RESEARCH INSTITUTE, BERKELEY, CALIFORNIA 94720

Current address: Department of Mathematics, The College of Charleston, Charleston, South Carolina 29424