THE CONJUGACY PROBLEM FOR GRAPH PRODUCTS WITH INFINITE CYCLIC EDGE GROUPS

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ABSTRACT. Finite graph products of groups with solvable conjugacy problem and infinite cyclic edge groups are considered. It is shown that the graph product has solvable conjugacy problem if the images of the edge group generators in each vertex group G_v are powers of a common central element c , where the group generated by c has solvable generalized word problem in G_v .

1. INTRODUCTION

It is known that HNN-extensions and free products with amalgamation of groups with a solvable conjugacy problem may have unsolvable conjugacy problem. For example in [6], C. F. Miller III constructed an HNN-extension G of a free group such that G has unsolvable conjugacy problem. Some restrictions placed on these constructions force the conjugacy problem to be solvable. This type of problem has been considered by R. D. Hurwitz [3] and S. Lipschutz [4], among others. The conjugacy problem for the more general construction of graph products (the fundamental group of a graph of groups as defined by Bass and Serre—see Serre [7]) has been studied by K. J. Horadam [1,2]. Horadam has shown that finite graph products with finite cyclic edge groups inherit a solvable conjugacy problem from their vertex groups under certain conditions. Horadam conjectured that the same is true of finite graph products with infinite cyclic edge groups. In this paper, we prove that conjecture.

2. NOTATION AND DEFINITIONS

Let D be a finite connected directed graph with edge set E , vertex set V , and source and terminus maps s and t and such that $e \in E$ implies $\bar{e} \in E$, where $\bar{e}$ is the inverse edge of e . Let (G, D) be a graph of groups; that is, for each $v \in V$, there is a vertex group G_v and for each $e \in E$, there is an edge group G_e and $G_e = G_{\bar{e}}$. Further, for each $e \in E$, there are partially recursive monomorphisms $A_e: G_e \rightarrow G_{se}$ and $B_e: G_e \rightarrow G_{te}$ and $A_e = B_{\bar{e}}$.

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Let T be a maximal tree of D . Then the graph product G^* has presentation $G^* = \langle G_v (v \in V), e (e \in E); \bar{e} = e^{-1}; eB_e(x)e^{-1} = A_e(x) (x \in G_e), e = 1 (e \in T) \rangle$.

We consider graph products such that all groups involved are recursively presented and for each $e \in E$, G_e is an infinite cyclic group $\langle k_e \rangle$. As defined by Horadam, for $v \in V$, let $H_v \subset G_v$ be

$$H_v = \{A_e(k_e) : v = se, e \in E\} \cup \{B_e(k_e) : v = te, e \in E\}.$$

We prove the following theorem.

Theorem. *If D is a nonempty finite connected graph, if (G, D) is a graph of (recursively presented) groups, and if*

- (i) $G_e = \langle k_e \rangle$ is infinite cyclic for all $e \in E$;
- (ii) G_v has solvable conjugacy problem for all $v \in V$;
- (iii) there exist $c_v \in G_v$ such that

$$H_v \subset \langle c_v \rangle \subset Z(G_v), \text{ the center of } G_v, \text{ for all } v \in V;$$

(iv) $\langle c_v \rangle$ has solvable generalized word problem in G_v for all $v \in V$, then G^* has solvable conjugacy problem.

3. LEMMAS AND PROOFS

In [1] Horadam showed that under the hypotheses of the theorem the conjugacy problem is solvable for any two elements of G^* of which at least one has nonzero reduced length. She further found a sufficient condition for the conjugacy problem to be solvable for two elements of zero reduced length. We state this sufficient condition in slightly different terms.

Suppose that pr is a function from E to ordered pairs of nonzero integers such that $\text{pr}(e) = (a, b)$ implies $\text{pr}(\bar{e}) = (b, a)$. Given a path $\pi = e_1 \cdots e_n$ in D , let $\text{pr}(e_i)$ be denoted (a_i, b_i) for $i = 1, \dots, n$. For nonzero integers α and β , define $\alpha \approx_\pi \beta$ to mean $\beta = (\alpha b_1 \cdots b_n) / (a_1 \cdots a_n)$ and $a_1 | \alpha, a_2 | (\alpha b_1 / a_1), \dots, a_n | ((\alpha b_1 \cdots b_{n-1}) / (a_1 \cdots a_{n-1}))$. (Note that this is equivalent to $\alpha = (k a_1 G_1(\pi)) / G(\pi)$ and $\beta = (k b_n G_n(\pi)) / G(\pi)$ for some integer k , where $G_i(\pi)$ and $G(\pi)$ are as defined by Horadam in [1].) Let

$$C(v, w) = \{(\alpha, \beta) : \alpha \approx_\pi \beta \text{ for some path } \pi \text{ from } v \text{ to } w\}.$$

In [1] Horadam showed that under assumptions (i)–(iv) of the theorem, with the particular pr given by $\text{pr}(e) = (a, b)$ where $A_e(k_e) = c_{se}^a$ and $B_e(k_e) = c_{te}^b$, G^* has solvable conjugacy problem for elements of zero reduced length if and only if $C(v, w)$ is uniformly recursive for all v and w in V . Since D is finite, this is equivalent to $C(v, w)$ being recursive for all v and w in V .

The following lemma is immediate from the definition of $\approx$.

Lemma 1. *If π_1 is a path such that $\alpha \approx_{\pi_1} \sigma$ and π_2 is a path such that $\sigma \approx_{\pi_2} \beta$ and if $t(\pi_1) = s(\pi_2)$, then $\alpha \approx_{\pi_1 \pi_2} \beta$.*

Let v and w be vertices in D and pr be as above; let $p_0, \dots, p_m$ be a list of all prime divisors of either coordinate of any $\text{pr}(e)$ for $e \in E$ and define $T(v, w) = \{(\alpha, \beta) : (\alpha, \beta) \in C(v, w), \alpha \text{ and } \beta \text{ are products of } p_0, \dots, p_m \text{ only}\}$.

Lemma 2. $C(v, w)$ is recursive if and only if $T(v, w)$ is recursive.

Proof. Given nonzero integers α and β , $\alpha = \alpha'\alpha''$ and $\beta = \beta'\beta''$ where α' and β' have only $p_0, \dots, p_m$ as prime factors and α'' and β'' are $p_0, \dots, p_m$ free. It is easy to see that $(\alpha, \beta) \in C(v, w)$ if and only if $\alpha'' = \beta''$ and $(\alpha', \beta') \in T(v, w)$ or $\alpha'' = -\beta''$ and $(\alpha', -\beta') \in T(v, w)$. Hence, the lemma follows. $\square$

Lemma 3. Let D be a finite graph with $\bar{e} \in E$ if $e \in E$, and let pr be as defined above. Then for all v and w in V , $T(v, w)$ is recursive and so $C(v, w)$ is also recursive.

Proof. Suppose $V = \{v_1, \dots, v_k\}$. Let $p_0, \dots, p_m$ be as before. Let T be $\pm$ all products of $p_0, \dots, p_m$. Define a function $f: T \rightarrow$ words on $\{x_0, \dots, x_m, -1\}$ by $f(p_0^{j_0} \dots p_m^{j_m}) = x_0^{j_0} \dots x_m^{j_m}$ and $f(-p_0^{j_0} \dots p_m^{j_m}) = (-1)x_0^{j_0} \dots x_m^{j_m}$. Define a semigroup S with identity 1 as follows.

generators: $x_0, \dots, x_m, v_1, \dots, v_k, -1$;

defining relations:

$$(*) \quad f(a)v = f(b)w, \quad \text{where } se = v, te = w, \text{ and } \text{pr}(e) = (a, b), \quad e \in E,$$

$$x_i x_j = x_j x_i, \quad v_s v_t = v_t v_s, \quad x_i v_s = v_s x_i,$$

$$(-1)x_i = x_i(-1), \quad (-1)v_s = v_s(-1),$$

$$(-1)(-1) = 1,$$

$$\text{for } i, j = 0, \dots, m \text{ and } s, t = 1, \dots, k.$$

Claim. For $\alpha, \beta \in T$, $\alpha \approx_\pi \beta$ for some path π in D from v to w if and only if $f(\alpha)v = f(\beta)w$ in S .

($\Rightarrow$) Suppose $\pi = e_1 \dots e_d$, relabeling the edges for ease of notation. The proof is by induction on d . If $d = 1$ then $\beta = \alpha b_1/a_1$ and $a_1 | \alpha$. So $f(\beta)w = f(\alpha/a_1)f(b_1)w = f(\alpha/a_1)f(a_1)v = f(\alpha)v$ in S . If $d > 1$, then $\alpha \approx_{e_1} \alpha b_1/a_1$ and $\alpha b_1/a_1 \approx_{e'} \beta$, where $e' = e_2 \dots e_d$. By the induction hypothesis, $f(\alpha)v = f(\alpha b_1/a_1)v'$ and $f(\alpha b_1/a_1)v' = f(\beta)w$ in S where $v' = s(e_2) = t(e_1)$. Therefore, $f(\alpha)v = f(\beta)w$ in S .

($\Leftarrow$) Now suppose that $f(\alpha)v = f(\beta)w$ in S . Then there is a sequence of equalities

$$f(\alpha)v = (-1)^{\sigma_1} X_1 Y_1 = (-1)^{\sigma_2} X_2 Y_2 = \dots = (-1)^{\sigma_d} X_d Y_d = f(\beta)w,$$

where X_i is a word on $\{x_0, \dots, x_m\}$, Y_i is a word on $\{v_1, \dots, v_k\}$, $\sigma_j = 0$ or 1 for $j = 1, \dots, d$, and each step is the result of one application of a relation $(*)$ together with any necessary commuting relations. Since these relations cannot change the number of vertex symbols involved, each Y_i is a single vertex symbol.

The proof is by induction on the length of the sequence. If the sequence is of length one, then $f(\alpha)v = f(\beta)w$ is immediate from one application of a relation $(*)$. Therefore, the equations $f(\alpha)v = f(k)f(a)v$ and $f(\beta)w = f(k)f(b)w$ follow from the commuting relations alone for some nonzero integer k and for some $(a, b) = \text{pr}(e)$ such that e goes from v to w . Since $a | \alpha$ and $\beta = kb = \alpha b/a$, it follows that $\alpha \approx_e \beta$.

Now suppose the sequence is of length > 1 . By comments above, the sequence looks like

$$f(\alpha)v = f(\alpha_1)v_1 = f(\alpha_2)v_2 = \dots = f(\alpha_d)v_d = f(\beta)w.$$

By the case above, we know that $\alpha \approx_e \alpha_1$ where e goes from v to v_1 . By the induction hypothesis, there is a path π from v_1 to w such that $\alpha_1 \approx_\pi \beta$. By Lemma 1, $\alpha \approx_{e\pi} \beta$, completing the proof of the claim.

Since every finitely presented commutative semigroup has solvable word problem, S has solvable word problem and so $T(v, w)$ is recursive as claimed. $\square$

Hence, the theorem follows.

This theorem together with Horadam's result for graph products with finite cyclic edge groups have as a corollary Horadam's conjecture of [1] in which condition (i) of the above theorem is replaced by (i') $G_e = \langle k_e \rangle$ for all e in E . That is, the theorem remains true if the edge groups are cyclic of arbitrary order.

In [5] the author gave an example of a recursively presented group H with elements x and y of infinite order such that $\langle x \rangle$ and $\langle y \rangle$ both have solvable generalized word problem in H , H has solvable conjugacy problem, and the HNN-extension $\langle H, t; t^{-1}xt = y \rangle$ has unsolvable conjugacy problem. Since HNN-extensions are graph products, this shows that if hypothesis (iii) of the theorem is eliminated, the result does not hold.

It would be interesting to know what happens if one eliminates or weakens hypothesis (iii) of the theorem in the case where all groups involved are finitely presented.

Conjectures. (1) There is a nonempty finite connected graph D and graph of groups (G, D) such that G_e is infinite cyclic for all $e \in E$, $A_e(G_e)$ and $B_e(G_e)$ have solvable generalized word problem in G_{se} and G_{te} , respectively, G_v is finitely presented and has solvable conjugacy problem for all $v \in V$, and the graph product G^* has unsolvable conjugacy problem.

(2) There is a finitely presented group H with solvable conjugacy problem and an HNN-extension $G = \langle H, t; t^{-1}xt = y \rangle$ where x and y are elements of H of infinite order such that $\langle x \rangle$ and $\langle y \rangle$ both have solvable generalized word problem in H and G has unsolvable conjugacy problem.

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