FOURIER SUMMATION WITH KERNELS DEFINED BY JACOBI POLYNOMIALS

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(Communicated by J. Marshall Ash)

ABSTRACT. Trigonometric polynomial kernels defined by Jacobi polynomials are investigated. They generalize the classical Dirichlet kernel and the Fejér kernel. The asymptotic behavior of the corresponding Fourier summation obtained is leading to optimal kernels.

The Dirichlet kernel and the Fejér kernel are basic tools in the theory of classical Fourier expansions. We introduce trigonometric polynomial kernels defined by Jacobi polynomials. One class consists of kernels with alternating signs containing the Dirichlet kernel as a special case; the other class consists of nonnegative kernels containing the Fejér kernel as a special case. Therefore, these kernels are certain parameterized generalizations of the Dirichlet kernel resp. Fejér kernel. We investigate their asymptotic behavior, finding in this way optimal kernels, where optimality is defined with respect to the generalized Lipschitz condition. Throughout this work our main reference for notation and results concerning Fourier series is [4]. For other uses of Jacobi polynomials as summability kernels we refer to [2, 6].

1. GENERALIZATIONS OF THE DIRICHLET KERNEL

Fix $\lambda > 0$. We study the following row-finite θ factors, which are strongly related to certain Jacobi polynomials: For $n \in \mathbb{N}_0$, $k = 0, \dots, n$, set

$$(1) \quad a_{n,k}^\lambda = \frac{(\lambda)_{n+k}(\lambda)_{n-k}(n!)^2}{(n+k)!(n-k)!((\lambda)_n)^2} \quad \text{and} \quad a_{n,-k}^\lambda = a_{n,k}^\lambda,$$

and for $t \in]-\pi, \pi]$, denote

$$(2) \quad D_n^\lambda(e^{it}) = \sum_{k=-n}^n a_{n,k}^\lambda e^{ikt}.$$

For $\lambda = 1$ we have the classical Dirichlet kernel.

Proposition 1. *We have*

$$(3) \quad D_n^\lambda(e^{it}) = b_n^\lambda R_n^{(\lambda-1/2, 1/2)}(\cos t),$$

Received by the editors May 30, 1990 and, in revised form, October 1, 1990.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 42A10.

Key words and phrases. Fourier approximation, Jacobi polynomials.

where $R_n^{(\lambda, 1/2, -1/2)}(x)$ is the n th Jacobi polynomial to the parameters $\alpha = \lambda - \frac{1}{2}$, $\beta = -\frac{1}{2}$, see [7], normalized such that $R_n^{(\lambda-1/2, -1/2)}(1) = 1$, and where

$$(4) \quad b_n^\lambda = \frac{(n!)^2 (2\lambda)_{2n}}{((\lambda)_n)^2 (2n)!}.$$

Proof. Formulas (2.7), (3.15), and Gegenbauer's formula (7.5) in [3] yield for $\lambda > -\frac{1}{2}$:

$$\begin{aligned} R_{2n}^{(\lambda-1/2, \lambda-1/2)}(\cos \frac{t}{2}) &= \frac{(2n)!}{(2\lambda)_{2n}} \sum_{k=0}^{2n} \frac{(\lambda)_{2n-k} (\lambda)_k}{(2n-k)! k!} \cos(n-k)t \\ &= \frac{(2n)!}{(2\lambda)_{2n}} \sum_{k=-n}^n \frac{(\lambda)_{n+k} (\lambda)_{n-k}}{(n+k)! (n-k)!} e^{ikt}. \end{aligned}$$

Using (3.13) in [3], we have

$$R_n^{(\lambda-1/2, -1/2)}(\cos t) = \frac{(2n)!}{(2\lambda)_{2n}} \sum_{k=-n}^n \frac{(\lambda)_{n+k} (\lambda)_{n-k}}{(n+k)! (n-k)!} e^{ikt},$$

which gives identity (3). $\square$

Remark. We have chosen the normalization $R_n^{(\lambda-1/2, -1/2)}(1) = 1$, since for $\lambda > 0$ it follows that $|R_n^{(\lambda-1/2, -1/2)}(x)| \leq 1$ for every $x \in [-1, 1]$, see [7, (7.33.1)]. Hence, a uniform bound for $|D_n^\lambda(e^{it})|$ is b_n^λ .

Theorem 1. For $\lambda > 1$ we have

- (i) $\lim_{n \rightarrow \infty} a_{n,k}^\lambda = 1$ for all $k \in \mathbb{N}_0$ and
- (ii) $\|D_n^\lambda\|_1 \leq M$ for all $n \in \mathbb{N}_0$.

Proof. Statement (i) follows from the asymptotic properties of the Gamma function. In fact, we have

$$a_{n,k}^\lambda = \frac{\Gamma(\lambda + n + k) \Gamma(\lambda + n - k) (\Gamma(1 + n))^2}{\Gamma(1 + n + k) \Gamma(1 + n - k) (\Gamma(\lambda + n))^2}.$$

Hence by [1, 6.1.46], we obtain for fixed $k \in \mathbb{N}_0$

$$(5) \quad \lim_{n \rightarrow \infty} a_{n,k}^\lambda = \lim_{n \rightarrow \infty} \frac{(n+k)^\lambda (n-k)^\lambda n^2}{(n+k)(n-k)n^{2\lambda}} = 1.$$

Statement (ii) follows from asymptotic properties of Jacobi polynomials. Note that

$$\|D_n^\lambda\|_1 = \frac{b_n^\lambda}{\pi} \int_0^\pi |R_n^{(\lambda-1/2, -1/2)}(\cos t)| dt.$$

By [7, Theorem 8.21.13], we have for $t \in [\frac{1}{n}, \pi - \frac{1}{n}]$

$$R_n^{(\lambda-1/2, -1/2)}(\cos t) = \frac{n!}{(\lambda + 1/2)_n n^{1/2}} k(t) (\cos(Nt + \gamma) + (n \sin t)^{-1} O(1))$$

where $N = n + \lambda/2$, $\gamma = -\lambda\pi/2$ and

$$k(t) = \frac{1}{\pi^{1/2}} \frac{1}{(\sin t/2)^\lambda}.$$

Using [1, 6.1.46] again

$$\lim_{n \rightarrow \infty} \frac{1}{n} b_n^\lambda = \frac{(\Gamma(\lambda))^2}{\Gamma(2\lambda)} 2^{2\lambda-1}$$

follows. Hence, integrating over $[0, \frac{1}{n}]$, $[\frac{1}{n}, \pi - \frac{1}{n}]$ and $[\pi - \frac{1}{n}, \pi]$, we obtain (6)

$$\|D_n^\lambda\|_1 \leq M_1 + M_2 \frac{b_n^\lambda n!}{(\lambda + 1/2)_n n^{1/2}} \int_{1/n}^{\pi-1/n} \frac{dt}{(\sin t/2)^\lambda} \leq M_1 + M_3 \frac{1}{n^{\lambda-1}} \int_{1/n}^{\pi-1/n} \frac{dt}{t^\lambda},$$

with constants $M_1, M_2, M_3 \geq 0$ not depending on n , and (ii) follows. $\square$

As is known, Theorem 1 yields (cf. [4, Proposition 1.3.3]) the following result.

Corollary 1. *Let $\lambda > 1$. Denoting by X one of the Banach spaces $L^p(T)$, $1 \leq p < \infty$, or $C(T)$, we have for each $f \in X$:*

$$(iii) \quad \lim_{n \rightarrow \infty} \|D_n^\lambda * f - f\|_X = 0.$$

Remark. For $X = L^1(T)$ and $X = C(T)$ condition (iii) also implies property (i) and (ii), cf. [4, Theorem 1.3.5].

Further we can derive that $(D_n^\lambda)_{n \in \mathbb{N}_0}$ is saturated in X with order $O(n^{-2})$. In fact, by $\Gamma(n+a)/\Gamma(n+b) = n^{a-b}(1 + c_1/n + c_2/n^2 + \dots)$ (compare [1, 6.1.47]), we obtain

$$(7) \quad \lim_{n \rightarrow \infty} \frac{a_{n,k}^\lambda - 1}{n^{-2}} = -(\lambda - 1)k^2.$$

In particular, Proposition 12.1.2 of [4] implies

Corollary 2. *Let $\lambda > 1$. If $\|D_n^\lambda * f - f\|_X = o(n^{-2})$ with $n \rightarrow \infty$, then f is constant almost everywhere.*

Next we study the order of approximation by $D_n^\lambda * f$ in comparison with the order of best approximation. Integrating from 0 to $\frac{1}{n}$, we have for $\alpha > 0$

$$\int_0^{1/n} t^\alpha |D_n^\lambda(e^{it})| dt \leq M b_n^\lambda \int_0^{1/n} t^\alpha dt = O(n^{-\alpha}).$$

In the same way,

$$\int_{\pi-1/n}^\pi t^\alpha |D_n^\lambda(e^{it})| dt = O(n^{-\alpha})$$

holds with $n \rightarrow \infty$. According to (6),

$$\int_{1/n}^{\pi-1/n} t^\alpha |D_n^\lambda(e^{it})| dt \leq M \frac{1}{n^{\lambda-1}} \int_{1/n}^{\pi-1/n} t^\alpha \frac{1}{t^\lambda} dt.$$

Hence,

$$\int_{1/n}^{\pi-1/n} t^\alpha |D_n^\lambda(e^{it})| dt = \begin{cases} O(n^{-\alpha}) & \text{for } 0 < \alpha < \lambda - 1, \\ O(n^{-(\lambda-1)} \ln n) & \text{for } \alpha = \lambda - 1, \\ O(n^{-(\lambda-1)}) & \text{for } \alpha > \lambda - 1. \end{cases}$$

Combining this with [4, Proposition 1.6.3], we obtain

Corollary 3. Let $\lambda > 1$. If $f \in \text{Lip}^*(X; \alpha)$, it follows that

$$\|D_n^\lambda * f - f\|_X = \begin{cases} O(n^{-\alpha}) & \text{for } 0 < \alpha < \lambda - 1, \\ O(n^{-(\lambda-1)} \ln n) & \text{for } \alpha = \lambda - 1, \\ O(n^{-(\lambda-1)}) & \text{for } \alpha > \lambda - 1. \end{cases}$$

Therefore, $D_n^\lambda * f$ behave for $\lambda > 3$ asymptotically as does the element of best approximation (cf. [4, Corollary 2.2.2]).

2. GENERALIZATIONS OF THE FEJÉR KERNEL

In this section we consider positive sums of the trigonometric polynomials $A_k^\lambda(e^{it})$. First note that

$$D_n^\lambda(e^{it}) = \frac{n!}{(\lambda)_n} \frac{P_n^{(\lambda-1/2, -1/2)}(\cos t)}{P_n^{(-1/2, \lambda-1/2)}(1)},$$

where $P_n^{(\alpha, \beta)}(x)$ denotes n th Jacobi polynomial with the usual normalization $P_n^{(\alpha, \beta)}(1) = \frac{(\alpha+1)_n}{n!}$.

Fix $\lambda > 1$ and $0 \leq \mu \leq \lambda - 1$. For $n \in \mathbb{N}_0$, $k = 0, \dots, n$, set

$$(8) \quad b_{n,k}^{\lambda, \mu} = \frac{(\mu+1)_{n-k}(\mu+1)_k(\lambda)_k}{(n-k)!k!k!}.$$

Further, set $\beta_n^{\lambda, \mu} = \sum_{k=0}^n b_{n,k}^{\lambda, \mu}$ and denote

$$(9) \quad F_n^{\lambda, \mu}(e^{it}) = \frac{1}{\beta_n^{\lambda, \mu}} \sum_{k=0}^n b_{n,k}^{\lambda, \mu} D_k^\lambda(e^{it}).$$

Now using a profound result by Gasper [5] implies that $F_n^{\lambda, \mu}(e^{it}) \geq 0$ for all $t \in]-\pi, \pi]$. Note that for $\lambda = 1$ and $\mu = 0$, we have the classical Fejér kernel. Since for $\lambda > 1$ and $0 \leq \mu \leq \lambda - 1$ the kernel functions $F_n^{\lambda, \mu}$ are positive, we investigate their first Fourier coefficients. From (8), (9) and (1), (2) we obtain

$$(F_n^{\lambda, \mu})^\wedge(1) = \frac{1}{\beta_n^{\lambda, \mu}} \sum_{k=0}^n b_{n,k}^{\lambda, \mu} \frac{k(k+\lambda)}{(k+1)(k+\lambda-1)}.$$

We prove now that $\lim_{n \rightarrow \infty} (F_n^{\lambda, \mu})^\wedge(1) = 1$ is valid. In fact, given $\varepsilon > 0$, we choose $n_0 \in \mathbb{N}$ such that $1 - \frac{k(k+\lambda)}{(k+1)(k+\lambda-1)} < \varepsilon$ for each $k > n_0$. Thus, for $n \geq n_0$,

$$\begin{aligned} |1 - (F_n^{\lambda, \mu})^\wedge(1)| &\leq \frac{1}{\beta_n^{\lambda, \mu}} \sum_{k=0}^{n_0} b_{n,k}^{\lambda, \mu} \left(1 - \frac{k(k+\lambda)}{(k+1)(k+\lambda-1)}\right) + \varepsilon \\ &\leq \frac{1}{\beta_n^{\lambda, \mu}} \sum_{k=0}^{n_0} b_{n,k}^{\lambda, \mu} + \varepsilon. \end{aligned}$$

Therefore, we have to show that the $(1/\beta_n^{\lambda, \mu}) \sum_{k=0}^{n_0} b_{n,k}^{\lambda, \mu}$ tends to zero with $n \rightarrow \infty$. The asymptotic of the Gamma function yields

$$\sum_{k=0}^{n_0} b_{n,k}^{\lambda, \mu} = O(n^\mu) \quad \text{for } n \rightarrow \infty.$$

To determine the asymptotic order of $\beta_n^{\lambda, \mu}$ we prove the following auxiliary result, which is likely to be known.

Lemma 1. Assume that u_k and v_k , $k \in \mathbb{N}_0$, are two numerical sequences with $u_k = O(k^\alpha)$ and $v_k = O(k^\beta)$ for $k \rightarrow \infty$ with $\alpha, \beta > -1$. Then, defining

$$w_n = \sum_{k=0}^n u_{n-k} v_k,$$

we have $w_n = O(n^{\alpha+\beta+1})$ for $n \rightarrow \infty$.

Proof. Let $|u_k| \leq M_1 k^\alpha$ and $|v_k| \leq M_2 k^\beta$ for any $k \in \mathbb{N}_0$. Then

$$\begin{aligned} \left| \frac{w_n}{n^{\alpha+\beta+1}} \right| &\leq \frac{M_1 M_2}{n^{\alpha+\beta+1}} \sum_{k=1}^{n-1} (n-k)^\alpha k^\beta + \frac{1}{n^{\alpha+\beta+1}} (|u_n v_0| + |u_0 v_n|) \\ &= M_1 M_2 \sum_{k=1}^{n-1} \frac{1}{n} \left(1 - \frac{k}{n}\right)^\alpha \left(\frac{k}{n}\right)^\beta \\ &\quad + \frac{1}{n^{\alpha+\beta+1}} (|u_n v_0| + |u_0 v_n|) \rightarrow M_1 M_2 \int_0^1 (1-x)^\alpha x^\beta dx \end{aligned}$$

for $n \rightarrow \infty$. $\square$

Setting $u_k = (\mu+1)_k/k! = O(k^\mu)$ and $v_k = (\mu+1)_k(\lambda)_k/k!k! = O(k^{\mu+\lambda-1})$, Lemma 1 gives $\beta_n^{\lambda, \mu} = O(n^{2\mu+\lambda})$. Therefore, $\lim_{n \rightarrow \infty} (F_n^{\lambda, \mu})^\wedge(1) = 1$ holds, and according to [4, Proposition 1.3.10], we have the following result:

Theorem 2. Let $\mu \geq 0$, $\lambda \geq 1$; further, let X denote $L^p(T)$, $1 \leq p < \infty$, or $C(T)$. Then for all $f \in X$

$$\lim_{n \rightarrow \infty} \|F_n^{\lambda, \mu} * f - f\|_X = 0$$

is valid.

The rate of convergence for $F_n^{\lambda, \mu}$ can also be derived by means of Lemma 1. Setting

$$u_k = \frac{(\mu+1)_k}{k!} = O(k^\mu)$$

and

$$v_k = \frac{(\mu+1)_k(\lambda)_k}{k!k!} \left(1 - \frac{k(k+\lambda)}{(k+1)(k+\lambda-1)}\right) = O(k^{\mu+\lambda-3}),$$

we obtain

$$\beta_n^{\lambda, \mu} - \beta_n^{\lambda, \mu} (F_n^{\lambda, \mu})^\wedge(1) = O(n^{2\mu+\lambda-2}),$$

provided $\mu + \lambda > 2$. Since $\beta_n^{\lambda, \mu} = O(n^{2\mu+\lambda})$, we get

$$1 - (F_n^{\lambda, \mu})^\wedge(1) = O(n^{-2}),$$

if $\mu + \lambda > 2$. Thus (see [4, Propositions 1.5.11, 1.5.12]) we get

Corollary 4. Let $\mu \geq 0$, $\lambda \geq 1$, and $\mu + \lambda > 2$. The positive trigonometric polynomials $F_n^{\lambda, \mu}$ define a Zygmund approximation sequence. In particular, we have

$$\lim_{n \rightarrow \infty} \|F_n^{\lambda, \mu} * f - f\|_X = O(n^{-\alpha})$$

for all $f \in \text{Lip}^*(X, \alpha)$.

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