THE PRESERVATION OF THE SEMIPRIME GOLDIE PROPERTY BY STRONG SEMILATTICE SUMS

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ABSTRACT. Let R be a strong semilattice sum of rings R_α ($\alpha \in P$) where P is an m.u.-semilattice. When P is infinite, R is not a right Goldie ring; and when P is finite, R is semiprime right Goldie iff each R_α is semiprime right Goldie.

I. INTRODUCTION

Let P be a semilattice whose natural order is indicated by $<$. As in [1], for any α, β in P , β is said to be maximal under α if $\beta < \alpha$, and there is no other γ in P such that $\beta < \gamma < \alpha$. P is called an m.u.-semilattice if the following conditions hold for P .

(1) If $\alpha, \gamma \in P$ with $\gamma < \alpha$, then there exists β maximal under α such that $\gamma \leq \beta < \alpha$.

(2) The set $\{\beta \in P \mid \beta \text{ is maximal under } \alpha\}$ is finite for each $\alpha \in P$.

A ring R is called a strong semilattice sum of rings R_α ($\alpha \in P$) by [2, p. 1018; 3, p. 12] if there exists a family of homomorphisms $\varphi_{\alpha, \beta}: R_\alpha \rightarrow R_\beta$ ($\alpha, \beta \in P$; $\alpha \geq \beta$) such that

- (a) R is the disjoint union of rings R_α ($\alpha \in P$);
- (b) $\varphi_{\alpha, \alpha}$ is the identity map of R_α ;
- (c) if $\alpha > \beta > \gamma$, then $\varphi_{\alpha, \beta} \varphi_{\beta, \gamma} = \varphi_{\alpha, \gamma}$;
- (d) if $a \in R_\alpha$ and $b \in R_\beta$, then the multiplication in R is determined by $a \cdot b = a \varphi_{\alpha, \alpha\beta} \cdot b \varphi_{\beta, \alpha\beta} \in R_{\alpha\beta}$.

M. L. Teply, E. G. Turman, and A. Quesada have shown that, when P is an m.u.-semilattice, R is π -semisimple iff each R_α is π -semisimple in [1]. We want to know if this is also true for the semiprime Goldie property.

II. PRELIMINARIES

Throughout this paper we need the monomorphism $*$: $R_\beta \rightarrow R$ for each

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$\beta \in P$ defined in [1] as

$$a^* = a - \left[\sum_{|\Delta|=1} a\varphi_{\beta, \Delta'} \right] + \sum_{|\Delta|=2} a\varphi_{\beta, \Delta'} \\ - \left[\sum_{|\Delta|=3} a\varphi_{\beta, \Delta'} \right] + \cdots + (-1)^n a\varphi_{\beta, \beta_1\beta_2\cdots\beta_n},$$

where $a \in R_\beta$, Δ is a nonempty subset of $\{\beta_1, \beta_2, \dots, \beta_n\}$, which is the set of elements maximal under β , Δ' denotes the product (g.l.b.) of all the β_i in Δ , and $|\Delta|$ is the cardinality of Δ .

As shown in [1], if $\beta < \alpha$ or α and β are not related by $<$, then for all $a \in R_\alpha$ and $b \in R_\beta$, $a^*b = 0 = ba^*$; and $a^*b = a^*b^*$, $ba^* = b^*a^*$ if $\alpha = \beta$. Also, when A is a (resp. right or left) ideal of R_α , A^* , which is isomorphic to A , is a (resp. right or left) ideal of R . Therefore, whenever we mention a property of R_α (resp. $a_\alpha \in R_\alpha$), R_α^* (resp. $a_\alpha^* \in R_\alpha^*$) has the same property, and vice versa.

Lemma 1. *If $\alpha, \beta \in P$ and $\alpha \neq \beta$, then $a^*b^* = 0 = b^*a^*$, for all $a \in R_\alpha$ and $b \in R_\beta$.*

Proof. $\alpha \neq \beta$ means either one of them is larger than the other, say $\alpha > \beta$, or α and β are not related by $<$. Write $b^* = b + b_1 + b_2 + \cdots + b_n$, where $b_i = (-1)^i \sum_{|\Delta|=i} b\varphi_{\beta, \Delta'}$. Then $a^*b = ba^* = a^*b_i = b_i a^* = 0$ for all $i = 1, \dots, n$ since either $\Delta' < \alpha$ or Δ' and α are not related by $<$. Therefore, $a^*b^* = b^*a^* = 0$. $\square$

Lemma 2. *If T is a semiprime ring and there exists some $a \in T$ such that $aT = 0$, then $a = 0$.*

Proof. Consider the right ideal $\langle a \rangle$ of T generated by a . Since $\langle a \rangle = \langle \mathbb{Z}a + aT \rangle$ and $a^2 \in aT = 0$, we can get $(\langle a \rangle)^2 = 0$; i.e. $\langle a \rangle$ is nilpotent. But T is semiprime, so $a \in \langle a \rangle = 0$. $\square$

Lemma 3. *If T is a semiprime ring, T' is a two-sided ideal of T , and I is an essential right ideal of T , then $I' = I \cap T'$ is an essential right ideal of T' .*

Proof. Suppose there is a nonzero right ideal K of T' such that $K \cap I' = 0$. Then for all $0 \neq k \in K$, kT is a right ideal of T . Since T' is an ideal of T , we have $(kT)^2 \cap I = kTkT \cap I \subseteq kT' \cap I \subseteq K \cap I' = 0$, which implies that $(kT)^2 = 0$; i.e. kT is nilpotent. Hence, by Lemma 2, $k = 0$. But this is contradictory to the choice of k . Therefore I' is essential in T' . $\square$

Lemma 4. *If R_α is semiprime for each $\alpha \in P$ and I_β^* is an essential right ideal of R_β^* , then the right ideal $I = I_\beta^* + \sum_{\alpha \neq \beta} R_\alpha^*$ is essential in R .*

Proof. Suppose there is a nonzero right ideal K of R such that $K \cap I = 0$. Then for all $0 \neq k \in K$, $k = \sum_{i=1}^n k_{\alpha_i}$ with $k_{\alpha_i} \in R_{\alpha_i} \setminus 0$ and $\alpha_i \in P$ for all $i = 1, \dots, n$. Now choose a maximal element α_0 from the set $\{\alpha_1, \dots, \alpha_n\}$. Then $kR_{\alpha_0}^* = k_{\alpha_0}R_{\alpha_0}^* = k_{\alpha_0}^*R_{\alpha_0}^*$. If $\alpha_0 = \beta$, then

$$k_{\alpha_0}^*R_{\alpha_0}^* \cap I_\beta \subseteq K \cap I = 0;$$

and when $\alpha_0 \neq \beta$, $k_{\alpha_0}^*R_{\alpha_0}^* \cap R_{\alpha_0}^* = kR_{\alpha_0}^* \cap R_{\alpha_0}^* \subseteq K \cap I = 0$ because $R_{\alpha_0}^* \subseteq I = I_\beta^* + \sum_{\alpha \neq \beta} R_\alpha^*$. Therefore $k_{\alpha_0}^*R_{\alpha_0}^* = 0$ in both cases. And so $k_{\alpha_0}^* = 0$ by Lemma

2; i.e. $k_{\alpha_0} = 0$. But this contradicts the assumption that $k_{\alpha_0} \neq 0$. Hence I must be essential in R . $\square$

If P is a finite semilattice, we can always find minimal elements in P . Furthermore, there is only one minimal element in P , for if both α and β are minimal elements then $\alpha\beta \leq \alpha$, β implies that $\alpha\beta = \alpha = \beta$. Thus we denote the unique minimal element in P by 0 . Also, by the definition of $*$, we have $R_0 = R_0^*$ and $a_0 = a_0^*$ for all $a_0 \in R_0$.

The following lemmas are proved to be true when P is finite.

Lemma 5. $R = \bigoplus_{\alpha \in P} R_\alpha^*$.

This lemma is a special case of Theorem 3.7 in [3]. However, since the proof is not readily accessible, we still show our work here.

Proof. First, we have $\sum_{\alpha \in P} R_\alpha^* \subseteq R$ because each R_α^* is an ideal of R . Next, for any $a \in R$, $a = \sum_{\alpha \in P} a_\alpha$ with $a_\alpha \in R_\alpha$ for each α . Let M_1 be the set of maximal elements in P . Then for each $\beta \in M_1$, $a_\beta^* = a_\beta + a_1 + \cdots + a_l$ with $a_\beta \in R_\beta$ and $a_i = (-1)^i \sum_{|\Delta|=i} a_\beta \varphi_{\beta, \Delta'}$ implies that

$$a_\beta = a_\beta^* - a_1 - \cdots - a_l.$$

Hence $a = \sum_{\beta \in M_1} a_\beta^* + \sum_{\alpha \in P_1} b_\alpha$ with $b_\alpha \in R_\alpha$ and $P_1 = P \setminus M_1$. Therefore $a \in \sum_{\alpha \in P} R_\alpha^*$ iff $\sum_{\alpha \in P_1} b_\alpha \in \sum_{\alpha \in P} R_\alpha^*$. Since P is finite and $R_0 = R_0^*$, we can repeat the procedure for $\sum_{\alpha \in P_1} b_\alpha$ and finally, have $a \in \sum_{\alpha \in P} R_\alpha^*$. So by Lemma 1 and the fact that $R_\alpha^* \cap R_\beta^* = 0$ whenever $\alpha \neq \beta$, we have $R = \bigoplus_{\alpha \in P} R_\alpha^*$. $\square$

Lemma 6. a is regular in R iff $a = \sum_{\alpha \in P} a_\alpha^*$ where a_α^* is regular in R_α^* for each α in P .

Proof. $a \in R$ implies that $a = \sum_{\alpha \in P} a_\alpha^*$ with $a_\alpha^* \in R_\alpha^*$ for each α in P by Lemma 5. Suppose there is β in P such that a_β^* is not regular in R_β^* ; i.e. there exists $0 \neq b_\beta^* \in R_\beta^*$ such that $a_\beta^* \cdot b_\beta^* = 0$ (resp. $b_\beta^* \cdot a_\beta^* = 0$). Then $a \cdot b_\beta^* = a_\beta^* \cdot b_\beta^* = 0$ (resp. $b_\beta^* \cdot a = 0$) contradicts that a is regular in R .

On the other hand, if each a_α^* is regular in R_α^* and if there is $0 \neq b \in R$ such that $ab = 0$ (resp. $ba = 0$) where $a = \sum_{\alpha \in P} a_\alpha^*$, write $b = \sum_{\alpha \in P} b_\alpha^*$, then

$$0 = ab = \left[\sum_{\alpha \in P} a_\alpha^* \right] \cdot \left[\sum_{\alpha \in P} b_\alpha^* \right] = \sum_{\alpha \in P} (a_\alpha^* \cdot b_\alpha^*)$$

implies $a_\alpha^* \cdot b_\alpha^* = 0$ (resp. $b_\alpha^* \cdot a_\alpha^* = 0$) for each α ; i.e. every $b_\alpha^* = 0$, which is a contradiction. $\square$

III. MAIN RESULTS

Theorem 1. R is not a right Goldie ring when P is infinite.

Proof. Let $M = \{\alpha_1, \alpha_2, \alpha_3, \dots\}$ be an infinite subset of P , and let $I_n = \sum_{i=1}^n R_{\alpha_i}^*$ for each positive integer n . Since each $R_{\alpha_i}^*$ is an ideal of R , so is each I_n . Then, by Lemma 1, $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$ is an infinite ascending chain of annihilators in R . Hence R is not a right Goldie ring. $\square$

To show a ring is semiprime right Goldie, we are going to apply Goldie's Theorem [4, Chapter 7].

Theorem 2. *If P is finite, R is semiprime right Goldie iff each R_α is semiprime right Goldie.*

Proof. Assume R is semiprime right Goldie. For each $\beta \in P$, let I_β^* be any essential right ideal of R_β^* , and let $I = I_\beta^* + \sum_{\alpha \neq \beta} R_\alpha^*$. Then I is an essential right ideal of R by Lemma 4. Also, I contains a regular element a of R by Goldie's Theorem. By Lemma 6, we can write $a = \sum_{\alpha \in P} a_\alpha^*$ where each a_α^* is regular in R_α^* . Then $a_\beta^* \in I_\beta^*$ since both a and $\sum_{\alpha \neq \beta} a_\alpha^*$ are in I .

Since R itself is an essential right ideal, R has a regular element by Goldie's Theorem. So each R_α^* contains a regular element by Lemma 6. Let a_β^* be a regular element in R_β^* , choose any regular element a_α^* from R_α^* for each $\alpha \neq \beta$, and let $a = \sum_{\alpha \in P} a_\alpha^*$. Then a is regular in R . Hence aR is an essential right ideal of R by [4, Lemma 7.2.3]. Also, $a_\beta^* R_\beta^* = [\sum_{\alpha \in P} a_\alpha^* R_\alpha^*] \cap R_\beta^* = aR \cap R_\beta^*$ shows that $a_\beta^* R_\beta^*$ is an essential right ideal of R_β^* by Lemma 3. Whence R_β^* is, and so R_β is, semiprime right Goldie.

Conversely, suppose each R_α is semiprime right Goldie, and I is any essential right ideal of R . For each α , let $I_\alpha^* = I \cap R_\alpha^*$. $I_\alpha^* \neq 0$ because I is essential and R_α^* is a nonzero ideal in R . So I_α^* is an essential right ideal in R_α^* by Lemma 3. Then I_α^* contains a regular element a_α^* of R_α^* . Therefore $a = \sum_{\alpha \in P} a_\alpha^*$ is a regular element in R and is contained in I .

For each regular element a in R , $a = \sum_{\alpha \in P} a_\alpha^*$ with a_α^* regular in R_α^* for each α . Suppose there is a nonzero right ideal K of R such that $K \cap aR = 0$. Then for any nonzero $k \in K$, with $k = \sum_{\alpha \in P} k_\alpha^*$, $k_\beta^* R_\beta^* \cap a_\beta^* R_\beta^* \subseteq K \cap aR = 0$ implies $k_\beta^* = 0$ by Lemma 2 for each β , which contradicts the choice of a nonzero k . Hence aR is an essential right ideal of R . Therefore R is semiprime right Goldie. $\square$

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