ON THE PERIOD-ENERGY RELATION

LARRY BATES AND JEDRZEJ ŚNIATYCKI

(Communicated by Kenneth R. Meyer)

ABSTRACT. We give a new proof of the period-energy relation for Hamiltonian systems.

Suppose that a Hamiltonian system (P, ω, h) has a manifold of periodic solutions. This means that there is a free S^1 action and the trajectories have no isotropy. In other words, we have an S^1 fibre bundle where the fibre is the trajectory. With no loss of generality, we can assume that on P , X_h has only periodic solutions. Define an $S^1 = R/Z$ action Φ by

$$(1) \quad \Phi^t(p) = \Phi_h^{\tau t}(p).$$

Here Φ_h is the flow of the Hamiltonian and τ is the period of a trajectory. Now we may regard P as an S^1 principal bundle with the action Φ .

Suppose that j is the momentum map of the S^1 action Φ . Now $X_j = \tau X_h$ (this follows by differentiating equation (1)), and

$$dj = i_{X_j}\omega = i_{\tau X_h}\omega = \tau dh.$$

Hence $0 = d^2j = d\tau \wedge dh$. We conclude that τ is a function only of h . Hence the period depends only on the energy if the energy surface is connected.

The only thing we need to do is show that Φ really does have a momentum map. To this end, denote this infinitesimal generator of Φ_h by X , of Φ by Y , and let Z be a vector at the point p . Extend Z to a vector field, also denoted by Z . We can arrange things so that $\Phi_*^t Z = Z$, or infinitesimally, $L_Y Z = 0$. Let ζ be a one-form with $\omega = -d\zeta$ (it suffices to do this in a neighborhood of a trajectory). Define

$$j(p) = \int_0^1 \langle \zeta, Y \rangle \circ \Phi^t(p) dt.$$

Thus, (dropping the reference to p)

$$\langle dj, Z \rangle = \int_0^1 \langle d\langle \zeta, Y \rangle, Z \rangle \circ \Phi^t dt,$$

Received by the editors October 7, 1990.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 58F05, 70H05.

Research partially supported by NSERC grants OGP0042416 and A8091.

© 1992 American Mathematical Society
 0002-9939/92 \$1.00 + \$.25 per page

and since $d\langle \zeta, Y \rangle = L_Y \zeta - d\zeta(Y, \cdot)$ we have

$$\begin{aligned} \langle d\langle \zeta, Y \rangle, Z \rangle &= \langle L_Y \zeta, Z \rangle - d\zeta(Y, Z) \\ &= L_Y \langle \zeta, Z \rangle - \langle \zeta, L_Y Z \rangle - d\zeta(Y, Z). \end{aligned}$$

However,

$$\int_0^1 L_Y \langle \zeta, Z \rangle \circ \Phi^t dt = \int_0^1 \frac{d}{dt} \langle \zeta, Z \rangle \circ \Phi^t dt = 0,$$

and $L_Y Z = 0$, thus

$$\begin{aligned} \langle dj, Z \rangle &= - \int_0^1 d\zeta(Y, Z) \circ \Phi^t dt \\ &= -\tau \int_0^\tau d\zeta(X, Z) \circ \Phi_h^t dt. \end{aligned}$$

We show that the integrand is invariant under the flow Φ_h . Now

$$L_X(d\zeta(X, Z)) = (L_X d\zeta)(X, Z) + d\zeta(L_X X, Z) + d\zeta(X, L_X Z),$$

and $L_X d\zeta = 0$ since X is Hamiltonian, $L_X X = 0$, and $L_X Z = \langle d \ln \tau, Z \rangle X$. Hence

$$\langle dj, Z \rangle = -\tau d\zeta(X, Z) = \omega(Y, Z),$$

which is precisely what is required to show that j is a momentum map.

This relation can also be proved by using averaging, as in Gordon [1], or using integral invariants and Stokes' theorem as in Moser [2], or one can use canonical relations as in Weinstein [3].

REFERENCES

1. W. Gordon, *On the relation between period and energy in periodic dynamical systems*, J. Math. Mech. **19** (1969), 111–114.
2. J. Moser, *Regularization of Kepler's problem and the averaging method on a manifold*, Comm. Pure Appl. Math. **23** (1970), 609–636.
3. A. Weinstein, *Lagrangian submanifolds and Hamiltonian systems*, Ann. of Math. **98** (1973), 377–410.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALGARY, CALGARY, ALBERTA, CANADA
T2N 1N4