REQUIRED REDUNDANCY IN THE REPRESENTATION OF REALS

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ABSTRACT. Redundancy in decimal-like representations of reals cannot be avoided. It is proved here that if $\{A_i\}_{i=0,1,2,\dots}$ is a countable collection of countable (or finite) sets of reals such that for each real x there are $a_i \in A_i$ with

$$x = \sum_{i=0}^{\infty} a_i,$$

then there is a dense subset of reals with redundant representations; that is, there is a dense set C of $\mathbb{R}$ such that for each x in C , $x = \sum_{i=0}^{\infty} a_i$ and $x = \sum_{i=0}^{\infty} b_i$ with a_i, b_i in A_i , but $a_i \neq b_i$ for some i . Petkovsek [1] proved a similar result under the added assumption that every sum of the form $\sum_{i=0}^{\infty} a_i$ with $a_i \in A_i$ converges.

INTRODUCTION

Every real number can be expressed as a decimal expansion, and each decimal expansion is shorthand for the limit of a convergent series. Let $A_0 = \mathbb{Z}$, and for each $i = 1, 2, \dots$ let $A_i = \{0, 10^{-i}, 2 \times 10^{-i}, \dots, 9 \times 10^{-i}\}$. Then each real number can be written as $\sum_{i=0}^{\infty} a_i$ where $a_i \in A_i$. Sometimes it is annoying that certain numbers have two different representations as such infinite series, e.g., $.19999\dots = .2000\dots$.

One might hope that it would be possible to eliminate this annoying redundancy by choosing different sets of numbers A_i . For example, we might let $A_1 = \{\pi/4, e/7, 34, -21/3\}$, and let the other A_i 's be equally unlikely sets of reals including negative numbers and large numbers. We would insist only that every real number is the limit of some series $\sum_{i=0}^{\infty} a_i$ and a_i is in A_i .

The standard representation of numbers has the added property that for every choice of $a_i \in A_i$, $\sum_{i=0}^{\infty} a_i$ does in fact converge. Representations of reals in which each $\sum_{i=0}^{\infty} a_i$ with $a_i \in A_i$ converges were considered in [1]. Techniques in [1] show that if $\{A_i\}$ is a collection of finite sets of reals such that every $\sum_{i=0}^{\infty} a_i$ with $a_i \in A_i$ converges, and every real in some interval is of the form $\sum_{i=0}^{\infty} a_i$ with $a_i \in A_i$, then the set of numbers with redundant representations is dense in the interval. Basically the proof is to note that the function $f: \prod A_i \rightarrow \mathbb{R}$ defined by $f(a_0, a_1, \dots) = \sum_{i=0}^{\infty} a_i$ is a continuous function from a Cantor

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set over $\mathbb{R}$. If it were one-to-one any interval, then that interval would be homeomorphic to a subset of a Cantor set, which is not true.

In fact, even without the assumption that every $\sum_{i=0}^{\infty} a_i$ with $a_i \in A_i$ converges, it remains true that reals with redundant representations are dense.

Our theorem is the following.

Theorem. *Let $\{A_i\}_{i=0,1,2,\dots}$ be a countable collection of countable (or finite) sets of reals such that for each real x there are $a_i \in A_i$ with $x = \sum_{i=0}^{\infty} a_i$. Then there is a dense subset of reals with redundant representations, that is, there is a dense set C of $\mathbb{R}$ such that for each x in C , $x = \sum_{i=0}^{\infty} a_i$ and $x = \sum_{i=0}^{\infty} b_i$, with a_i, b_i in A_i , but $a_i \neq b_i$ for some i .*

DEFINITIONS AND NOTATION

For each $i = 0, 2, \dots$, A_i is a finite or countably infinite set of real numbers.

An *A-segment* is a finite sequence $a_0, \dots, a_{n-1}$ with $a_i \in A_i$, $i = 0, 1, 2, \dots, n-1$. Its sum is $\sum_{i=0}^{n-1} a_i$, and its length is n . Sometimes we denote an *A-segment* and its sum by the same letter (usually P or Q).

An *A-series* is an infinite sequence $a_0, a_1, \dots$ with $a_i \in A_i$ for $i = 0, 1, 2, \dots$. Its sum is the infinite sum $\sum_{i=0}^{\infty} a_i$, which exists only when the series converges. Its length is infinite. The assumption in the theorem is that every real number is the sum of some *A-series*.

One *A-segment* or *A-series* Q is an *extension* of an *A-segment* P if length $Q > \text{length } P$, and the first length P elements of Q equal the corresponding elements of P .

In the proof of the theorem, we will construct two *A-segments* P and Q with the same length but different sums (so they must be distinct), but that have *A-series* extensions with equal sums x . Thus, x has two distinct *A-series* representations.

Denote by $T(N)$ the *Nth tail set*, which is the set of all limits of convergent series $\sum_{i=N}^{\infty} a_i$ with $a_i \in A_i$. A compact way to state the assumption of the theorem is $T(0) = \mathbb{R}$.

$T(N, \mu)$, the *Nth tails of μ -bounded deviation*, is the subset of $T(N)$ with representations whose partial sums obey

$$\left| \sum_{i=N}^M a_i \right| < \mu \quad \text{for all } M.$$

The tail sets are of interest because the sum of an *A-segment* P added to all the numbers in the tail set $T(\text{length } P)$ gives all the reals expressible by *A-series* extensions of P . This set of reals is called the *leaves* of P .

Similarly, the μ -leaves of P , the set of which is denoted $\text{Leaves}(P, \mu)$, are all reals equal to the sum of P added to all the numbers in $T(\text{length } P, \mu)$. $\text{Leaves}(P, \mu)$ play a crucial role in the proofs. The μ -boundedness gives control over series that we build, leading to convergence.

For a set D of real numbers, $\text{cl}(D)$ denotes the closure of D .

BASIC LEMMAS AND PROPOSITIONS

Our first lemma states that the μ -tails are not too sparse.

Lemma 1. Assume $T(0) = \mathbb{R}$. Given $\mu > 0$, there exists N such that for all $i \geq N$, $\text{cl}(T(i, \mu))$ contains an open interval (which depends on i).

Proof of Lemma 1. We prove Lemma 1 by writing $\mathbb{R}$ as a countable union of sets and then applying the Baire Category Theorem.

Each real number x is the sum of an A -series $\sum_{i=0}^{\infty} a_i$, so for each x there is a number $N(x)$ such that $|\sum_{i=m}^n a_i| < \mu/4$ for all $n \geq m > N(x)$. Thus for all $m > N(x)$, $x \in \text{Leaves}(P, \mu/4)$ for some A -segment P of length m . Thus for any N ,

$$\mathbb{R} = \bigcup_{k=N}^{\infty} \cup \{\text{Leaves}(P, \mu/4) : P \text{ is an } A\text{-segment of length } k\}.$$

Since there are only countably many A -segments of any given finite length, this double union is a countable union. By the Baire Category Theorem, for some k and some A -segment P of length k , $\text{cl}(\text{Leaves}(P, \mu/4))$ contains an open interval. Since $\text{Leaves}(P, \mu/4)$ is a translate of $T(k, \mu/4)$, $\text{cl}(T(k, \mu/4))$ contains an open interval. Since N is arbitrary in the union, there is a sequence $k_1 < k_2 < k_3 < \dots$ such that for each $i = 1, 2, \dots$, $\text{cl}(T(k_i, \mu/4))$ contains an open interval.

Let n be an integer with $n > k_1$. We now show that $\text{cl}(T(n, \mu))$ contains an open interval.

Let $y \in T(k_1, \mu/4)$. Then $y = \sum_{i=k_1}^{\infty} a_i$ with $a_i \in A_i$ where $|\sum_{i=k_1}^m a_i| < \mu/4$ for all $m \geq k_1$. Therefore $|\sum_{i=n}^m a_i| < \mu/2$ for any $m \geq n$. Choose j with $n < k_j$. Then for each $x \in T(k_j, \mu/4)$, $\sum_{i=n}^{k_j-1} a_i + x$ is a point in $T(n, 3\mu/4)$. Since $\text{cl}(T(k_j, \mu/4))$ contains an interval, $\text{cl}(T(n, 3\mu/4))$, and therefore, $\text{cl}(T(n, \mu))$ contains an interval.

Lemma 2. Assume $T(0) = \mathbb{R}$. Suppose P is an A -segment, $\mu > 0$, I is a nondegenerate closed interval, and

$$\text{cl}(\text{Leaves}(P, \mu)) \supset I.$$

Then given $\delta > 0$, there exists a μ -extension Q of P and a nondegenerate closed interval $J \subset I$ such that

$$\text{cl}(\text{Leaves}(Q, \delta)) \supset J.$$

Proof of Lemma 2. Let $\lambda = \min(|I|/3, \delta)$. By Lemma 1, there is an N such that for $i > N$, $\text{cl}(T(i, \lambda))$ contains a nondegenerate closed interval $K(i)$. Let x be in the middle third of I and in $\text{Leaves}(P, \mu)$. μ -extend P to Q such that Q approximates x sufficiently well so that Q 's sum is in the middle third of I , and length $Q > N$. Then $\text{cl}(\text{Leaves}(Q, \lambda))$ contains a translate J of $K(\text{length } Q)$ (and so surely $\text{cl}(\text{Leaves}(Q, \delta)) \supset J$). Since $\text{cl}(\text{Leaves}(Q, \lambda)) \subset I$, $J \subset I$.

Our next proposition is a key in deriving the fact that a number has two representations.

Proposition 1. Assume $T(0) = \mathbb{R}$. Let P and Q be A -segments with $\text{cl}(\text{Leaves}(P, 1))$ and $\text{cl}(\text{Leaves}(Q, 1))$ both containing an open interval I . Then there is a real number in $\text{Leaves}(P, 1) \cap \text{Leaves}(Q, 1)$.

Proof of Proposition 1. Using Lemma 2, extend P to P' such that $\text{cl}(\text{Leaves}(P', 1/2))$ contains a closed interval J in I . Then using the lemma again, extend Q to Q' such that $\text{cl}(\text{Leaves}(Q', 1/2))$ contains a closed interval I' in J . Thus both $\text{cl}(\text{Leaves}(P', 1/2))$ and $\text{cl}(\text{Leaves}(Q', 1/2))$ contain I' .

Now repeat the above steps, getting $\frac{1}{2}$ -extensions P'' , Q'' , and a closed subinterval I'' of I' , with $\text{cl}(\text{Leaves}(P'', 1/4))$ and $\text{cl}(\text{Leaves}(Q'', 1/4))$ both containing I'' . Continuing in this way gives a sequence of nested intervals converging to a point, and a sequence of further and further extensions P , P' , P'' , $\dots$ that form an A -series converging to that point. But Q , Q' , Q'' , $\dots$ also form an A -series converging to the same point. This completes the proof of Proposition 1.

PROOF OF THE THEOREM

The proof of the theorem divides into two cases, depending on whether or not there is a nonzero “offset.”

Definition. An *offset* is a number that is the limit of numbers $t_{n_i} \in T(n_i)$ where $n_1 < n_2 < n_3 < \dots$.

When a nonzero offset exists, the natural way to construct a series designed to converge to a certain value may yield a series that does not converge because it continually jumps. The following example illustrates this technical difficulty.

Example. Let

$$\begin{aligned} B_1 &= \{0, 0.1, 0.2, 0.3, \dots, 0.8\}, \\ B_2 &= \{0, -2\}, \\ B_3 &= \{0, 2.1\}, \\ B_4 &= \{0, 0.01, 0.02, 0.03, \dots, 0.08\}, \\ B_5 &= \{0, -2\}, \\ B_6 &= \{0, 2.01\}, \end{aligned}$$

and so on. Note that 1 is a limit point of the set of numbers to which convergent series with terms from the B_j 's converge, but 1 is not itself the sum of such a convergent series. The natural way of attempting to produce a series converging to 1 would contain infinitely many -2 's.

In this example, -2 is an offset, as is 2. In the ordinary decimal system where A_i is $\{0, 10^{-i}, 2 \times 10^{-i}, \dots, 9 \times 10^{-i}\}$, the diameter of the tails $T(i)$ converges to 0 as $i \rightarrow \infty$, so there is no offset except 0.

First we deal with the case where a nonzero offset exists.

Proposition 2. Assume that $T(0) = \mathbb{R}$. If there is an offset $d \neq 0$, then the set of real numbers that are sums of more than one A -series is dense in $\mathbb{R}$.

Proof. The aim of this proof is to construct two A -series with the same sum. One has partial sums that do not vary much after N terms, while the other incorporates a variation of approximately d .

Let I be an open interval. We will produce a point in I that is the sum of more than one A -series. We assume without loss of generality that the length of I is less than $|d|/2$.

Apply Lemma 1 with $\mu = |I|/3$. There exists L such that for $i > L$, $\text{cl}(T(i, \mu))$ contains an open interval.

Let Q be an A -segment of length $M > L$ whose sum is in the middle third of I . Then $\text{cl}(\text{Leaves}(Q, \mu))$ contains some interval $J \subset I$. Let y be the midpoint of J . By Lemma 1 there exists $M' > M$ such that for $i > M'$, $\text{cl}(T(i, |J|/3))$ contains an open interval.

Since $T(0) = \mathbb{R}$, there is an A -series P converging to $y - d$. Find an initial segment P^N of P of length $N > M'$ such that for $n \geq N$,

$$|P^n - (y - d)| < |J|/12.$$

We claim that P^N has a finite extension P' whose sum is in the middle third of J . To find one, use the definition of offset to find an $i > N$ and $d' \in T(i)$ such that $|d - d'| < |J|/12$. Let P' be the first i terms of P extended by sufficiently many terms of d' . Notice that $\text{cl}(\text{Leaves}(P', |J|/3))$ contains some interval $K \subset J$, so $\text{cl}(\text{Leaves}(P', \mu))$ and $\text{cl}(\text{Leaves}(Q, \mu))$ both contain K . Proposition 1 then implies that there is a number x that is a μ -leaf of both P' and Q . Since $x \in \text{Leaves}(Q, \mu)$, x is the sum of an A -series whose partial sums (of more than M terms) are all in I , unlike the sum of P^N , which is close to $(y - d)$ and is therefore not in I . Thus these two representations of x are different.

We now prove the theorem in the case where there is no nonzero offset.

Proposition 3. Assume that $T(0) = \mathbb{R}$. If there is no nonzero offset, then the set of real numbers that are sums of more than one A -series is dense in $\mathbb{R}$.

Proof. Let I be an open interval. We will produce a point in I that is the sum of more than one A -series.

Choose $\mu < |I|/6$. By Lemma 1 there is an integer N such that for $i > N$, $\text{cl}(T(i, \mu))$ contains an open interval. Let Q be an A -segment of length $M > N$ whose sum is in the middle third of I . Then $\text{cl}(\text{Leaves}(Q, \mu))$ contains a nondegenerate closed interval $J \subset I$.

We will now show that there are different μ -extensions Q' and Q'' of Q such that $\text{Leaves}(Q', 2\mu)$ and $\text{Leaves}(Q'', 2\mu)$ both contain the same single point x .

Consider the μ -leaves of Q that are in J and the corresponding A -series for each. For a given $i \geq \text{length } Q$, the set of these μ -leaves is naturally partitioned by the choice of $a \in A_i$ that appears in the series for a leaf. In other words, for each $a \in A_i$ we consider the subset P_a of the μ -leaves of Q in J that use a as the i th element in their series. Certainly there exists an $i \geq \text{length } Q$ such that this partition is nondegenerate, i.e., with 2 or more different P_a 's nonempty. For each such i , either there is only a finite number of P_a 's nonempty or an infinite number. We now consider separately two cases, depending on whether there are any i 's with infinitely many P_a 's nonempty.

Case 1. Suppose for every i , only finitely many P_a 's are nonempty. Take the first $i \geq \text{length } Q$ when 2 or more P_a 's are nonempty. Since only a finite number of P_a 's are nonempty,

$$J = \text{cl}\left(\bigcup\{P_a : a \in A_i\}\right) = \bigcup\{\text{cl}(P_a) : a \in A_i\};$$

so there are $a' \neq a''$, both in A_i , with $\text{cl}(P_{a'}) \cap \text{cl}(P_{a'') \neq \emptyset$ since J is an interval. Let x be a number in the intersection. Let Q' and Q'' denote 2 extensions of Q , both of length $i + 1$ (i.e., both ending with the i th term), using a' and

a'' , respectively, as their i th term. The point x , being in $\text{cl}(\text{Leaves}(Q, \mu))$, is in $\text{cl}(\text{Leaves}(Q', 2\mu))$ and $\text{cl}(\text{Leaves}(Q'', 2\mu))$.

Next we show that a standard diagonalization argument proves that x is actually in $\text{Leaves}(Q', 2\mu)$ and in $\text{Leaves}(Q'', 2\mu)$. The proof is the same for both, so we give it only for Q' . We build a candidate series and then show that it converges to x .

Find $y_j \in \text{Leaves}(Q', 2\mu)$ with y_j converging to x . Infinitely many of the y_j 's have the same $i+1$ term (more precisely, the representations that show that $y_j \in \text{Leaves}(Q', 2\mu)$ have the same $i+1$ term); so discard (for purposes of this paragraph only) any y_j 's with different $i+1$ term, and set b_1 to that $i+1$ term. Now, of the y_j 's remaining, infinitely many have the same $i+2$ term; so discard other y_j 's, and set b_2 to that $i+2$ term. Continuing in this way, we generate a sequence $b_j \in A_j$, $j > i$. Let P be the A -series that begins with Q' and then continues with the b_j 's. We now claim that P converges to x .

Since $y_j \in \text{Leaves}(Q', 2\mu)$, every partial sum (beyond length Q') of every y_j is between $Q' - 2\mu$ and $Q' + 2\mu$. Any partial sum of P is the partial sum of some y_j (actually of infinitely many y_j 's); so every partial sum of P also lies between $Q' - 2\mu$ and $Q' + 2\mu$. Therefore, any subsequence of the partial sums of P has a convergent subsequence. If any subsequence converged to $z \neq x$, then denoting the sum of the first n terms of a series by superscript n ,

$$P^{n_i} \rightarrow Sz,$$

and each P^{n_i} agrees with some y_j , call it y_i^* , through n_i terms. As a result,

$$y_i^* = y_i^{n_i} + t_i,$$

where $t_i \in T(n_i, 4\mu)$. $y_i^* \rightarrow x$, but $y_i^{n_i} \rightarrow z$, so $t_i \rightarrow x - z$, which contradicts the assumption that no nonzero offset exists. This concludes the proof of this case.

Case 2. Suppose that for some i , an infinite number of P_a 's are nonempty.

Then, since all of these a 's must be less than 2μ in absolute value, there is a cluster point of a 's. We know that $\text{cl}(T(i, \mu))$ contains some interval (Lemma 1) (which might be very short). Then $Q' + \text{cl}(T(i, \mu))$ contains some interval K , which must be in I (since μ is a small fraction of $|I|$). In Q' , replacing its i th term by two other a values that are closer together than the length of K , giving Q'' and Q''' , we have that

$$Q'' + \text{cl}(T(i, \mu)) \quad \text{and} \quad Q''' + \text{cl}(T(i, \mu))$$

both contain an interval in I , and the intervals overlap. By Proposition 1, there is a common μ -leaf.

This completes the proof of the theorem.

REFERENCES

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