LIFTING PRIMITIVITY OF FREE NILPOTENT GROUPS

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ABSTRACT. Let $F_{n,c}$ denote the free n -generator nilpotent group of class c . For $n \geq c + m - 1$, every primitive system of m elements of $F_{n,c}$ can be lifted to a primitive system of m elements of the free group F_n .

INTRODUCTION

Let $F = F_n$, $n \geq 2$, be a free group and V be a fully invariant subgroup of F . A system $\mathbf{w} = (w_1, \dots, w_m)$, $m \leq n$, $w_i \in F$, is said to be *primitive* in F if $\mathbf{w}$ can be extended to a basis of F , and $\mathbf{w}$ is said to be *primitive mod V* if the corresponding system $\bar{\mathbf{w}} = (\bar{w}_1, \dots, \bar{w}_m)$ of cosets $\bar{w}_i (= w_i V)$ can be extended to some basis of F/V . If $\mathbf{w} = (w_1, \dots, w_m)$ is primitive mod V and if there exists $\mathbf{v} = (v_1, \dots, v_m)$, $v_i \in V$, such that the corresponding system $\mathbf{w}^* = (w_1 v_1, \dots, w_m v_m)$ is primitive in F then we say that $\mathbf{w}$ *lifts* to $\mathbf{w}^*$. Lifting primitivity of free V -groups is a natural problem which is linked to the study of the automorphism groups $\text{Aut}(F/V)$. For $n \geq 4$, $m \leq n$, every primitive system $\mathbf{w} \bmod F''$ lifts to a primitive system $\mathbf{w}^*$ of F . This follows from the well-known result of Bachmuth and Mochizuki [3] that every automorphism of a free metabelian group of rank exceeding 3 is tame (i.e., induced by an automorphism of F). When F/V is free nilpotent of class exceeding two or free metabelian nilpotent of class exceeding two then there exist nontame automorphisms for arbitrary large values of n [1, 2, 4]. In [5] we studied the lifting of primitivity of free metabelian nilpotent groups and proved that for $n \geq m+2$, every primitive system $\mathbf{w} \bmod \gamma_{c+1}(F)F''$ lifts to a primitive system $\mathbf{w}^*$ of F . In this paper we study the corresponding problem of lifting primitivity of systems of elements in free nilpotent groups $F_{n,c} (\cong F/\gamma_{c+1}(F))$, $c \geq 2$. We prove that, for $n \geq c + m - 1$, every primitive system $\mathbf{w} \bmod \gamma_{c+1}(F)$ of m elements lifts to a primitive system $\mathbf{w}^*$ of F (Theorem C).

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PRELIMINARY LEMMAS

We use standard commutator notation: $[a, b] = a^{-1}b^{-1}ab$, $[a, b, c] = [[a, b], c]$, etc. Let $F = \langle x_1, x_2, \dots, x_n \rangle$ be free of rank n . We first prove some lemmas concerning the existence of certain specific automorphisms of F .

Lemma 1. *For any permutation π of the set $\{2, \dots, c\}$, $2 \leq c \leq n$, there is an automorphism θ_π of F of the form given*

$$\theta_\pi = \{x_1 \rightarrow x_1[x_1, x_{2\pi}, \dots, x_{c\pi}]v_1, x_i \rightarrow x_i u_i, \\ i = 2, \dots, c, x_j \rightarrow x_j, j \geq c+1\},$$

where $v_1 \in \gamma_{c+1}(F)$ and $u_i \in F'$ ($= \gamma_2(F)$).

Proof. For $c = 2$, $\theta = \theta_\pi = \{x_1 \rightarrow x_1[x_1, x_2], x_i \rightarrow x_i, i \neq 1\}$ is an automorphism of F of the required form. For $c \geq 3$, consider endomorphisms α, β of F given

$$\alpha = \{x_1 \rightarrow x_1[x_2, \dots, x_c], x_i \rightarrow x_i, i \neq 1\}, \\ \beta = \{x_2 \rightarrow x_2[x_2, x_1], x_i \rightarrow x_i, i \neq 2\}.$$

Then clearly α, β are IA-automorphisms of F (i.e., inducing identity modulo F'). Set $\theta = \alpha\beta\alpha^{-1}\beta^{-1}$. So,

$$\theta(x_2) \equiv x_2 \pmod{F'}, \quad \theta(x_i) = x_i \text{ for } i = 3, \dots, n,$$

and

$$\theta(x_1) \equiv x_1[x_1, x_2, \dots, x_c] \pmod{\gamma_{c+1}(F)}$$

(since $\beta^{-1}(x_1) = x_1$; $\alpha^{-1}\beta^{-1}(x_1) = x_1[x_2, \dots, x_c]^{-1}$; $\beta\alpha^{-1}\beta^{-1}(x_1) = x_1[x_2[x_2, x_1], x_3, \dots, x_c]^{-1} \equiv x_1[x_2, \dots, x_c]^{-1}[x_1, x_2, x_3, \dots, x_c] \pmod{\gamma_{c+1}(F)}$; $\alpha\beta\alpha^{-1}\beta^{-1}(x_1) \equiv x_1[x_1, x_2, x_3, \dots, x_c] \pmod{\gamma_{c+1}(F)}$). For any permutation π of the set $\{2, \dots, c\}$ define an automorphism θ_π of F by

$$\theta_\pi = \pi^* \theta \pi^{*-1}$$

where $\theta = \alpha\beta\alpha^{-1}\beta^{-1}$ is as above and π^* is given by

$$\pi^* = \{x_1 \rightarrow x_1, x_2 \rightarrow x_{2\pi}, \dots, x_c \rightarrow x_{c\pi}, x_j \rightarrow x_j, j \geq c+1\}.$$

Then it is clear that θ_π is of the required form:

$$\theta_\pi = \{x_1 \rightarrow x_1[x_1, x_{2\pi}, \dots, x_{c\pi}]v_1, x_i \rightarrow x_i u_i, \\ i = 2, \dots, c, x_j \rightarrow x_j, j \geq c+1\},$$

where $v_1 \in \gamma_{c+1}(F)$ and $u_i \in F'$.

Lemma 2. *Given $[x_1, y_1, \dots, y_{c-1}]$, $2 \leq c \leq n$, $y_i \in \{x_1, x_2, \dots, x_n\}$ ($y_1 \neq x_1$), there exists $\mu \in \text{Aut}(F)$ such that $\mu(x_1) \equiv x_1[x_1, y_1, \dots, y_{c-1}] \pmod{\gamma_{c+1}(F)}$; $\mu(x_i) \equiv x_i(F')$, $2 \leq i \leq c$, and $\mu(x_j) = x_j$, $j \geq c+1$.*

Proof. Let

$$\mathbf{z}(k) = (z_1, z_2, \dots, z_k) = (x_{i_1}, \dots, x_{i_k}), \\ 2 \leq i_1 < \dots < i_k \leq c, 1 \leq k \leq c-1,$$

be an arbitrary k -tuple of distinct generators of F and let

$$\mathbf{y}(k) = (y_1, y_2, \dots, y_k), y_i \in \{x_1, \dots, x_n\}, 1 \leq k \leq c-1,$$

be an arbitrary k -tuple.

Let $u(\mathbf{z}(k) \rightarrow \mathbf{y}(k))$ denote the commutator obtained from $u = [x_1, x_2, x_3, \dots, x_c]$ upon replacing the k -tuple $\mathbf{z}(k)$ by the k -tuple $\mathbf{y}(k)$. We prove by induction on $1 \leq k \leq c-1$ that there is an automorphism $\mu(\mathbf{z}(k) \rightarrow \mathbf{y}(k))$ of F such that

$\mu(\mathbf{z}(k) \rightarrow \mathbf{y}(k))x_1 \equiv x_1 u(\mathbf{z}(k) \rightarrow \mathbf{y}(k)) \pmod{\gamma_{c+1}(F)}$; $\mu(\mathbf{z}(k) \rightarrow \mathbf{y}(k))(x_k) \equiv x_i \pmod{F'}$, $2 \leq i \leq c$, and $\mu(\mathbf{z}(k) \rightarrow \mathbf{y}(k))(x_j) = x_j$, $j \geq c+1$. The statement of the lemma is the case $k = c-1$.

For $k = 1$, if $z_1 = y_1$ then there is nothing to prove. Otherwise, we consider

$$\alpha = \{z_1 \rightarrow z_1 y_1, x_j \rightarrow x_j, x_j \neq z_1\} \in \text{Aut}(F)$$

and, by Lemma 1, the automorphism θ of F given by

$$\theta = \{x_1 \rightarrow x_1[x_1, x_2, \dots, x_c]v_1, x_i \rightarrow x_i u_i, \\ i = 2, \dots, c, x_j \rightarrow x_j, j \geq c+1\},$$

where $v_1 \in \gamma_{c+1}(F)$, $u_i \in F'$. We define

$$\mu(\mathbf{z}(1) \rightarrow \mathbf{y}(1)) = \alpha^{-1} \theta \alpha \theta^{-1}$$

and verify instantly that it is of the required form.

For $k = 2, \dots, c-1$, assume the result for $k-1$.

Consider $\mathbf{y}(k) = (y_1, y_2, \dots, y_k)$ and define for each $t = 1, \dots, k$,

$$\mathbf{y}(k; t) = (y_1, \dots, y_{t-1}, z_t, y_{t+1}, \dots, y_k).$$

Then by the induction hypothesis there exist automorphisms $\mu(\mathbf{z}(k), \mathbf{y}(k; t))$, $t = 1, \dots, k$, such that $\mu(\mathbf{z}(k) \rightarrow \mathbf{y}(k; t))(x_1) \equiv x_1 u(\mathbf{z}(k) \rightarrow \mathbf{y}(k; t)) \pmod{\gamma_{c+1}(F)}$; $\mu(\mathbf{z}(k) \rightarrow \mathbf{y}(k; t))(x_i) \equiv x_i \pmod{F'}$, $2 \leq i \leq c$, and $\mu(\mathbf{z}(k) \rightarrow \mathbf{y}(k; t))(x_j) = x_j$, $j \geq c+1$.

In particular, if $z_t = y_t$ for some $t = 1, \dots, k$ then, for that value of t , $\mu(\mathbf{z}(k) \rightarrow \mathbf{y}(k)) = \mu(\mathbf{z}(k) \rightarrow \mathbf{y}(k; t))$ is an automorphism of the required form. Thus we may assume that $z_t \neq y_t$ for all $t = 1, \dots, k$. Further, if $z_t \in \{y_1, \dots, y_{t-1}, y_{t+1}, \dots, y_k\}$ for each $t = 1, \dots, k$, then $\mathbf{y}(k)$ is a permutation of $\mathbf{z}(k)$ and by Lemma 1 there exists an automorphism of the required form. Thus we may assume, without loss of generality, that $z_t \notin \{y_1, \dots, y_k\}$ for some $1 \leq t \leq k$. Define

$$\mu(\mathbf{z}(k) \rightarrow \mathbf{y}(k)) = \alpha_t^{-1} \mu(\mathbf{z}(k) \rightarrow \mathbf{y}(k; t)) \alpha_t \mu(\mathbf{z}(k) \rightarrow \mathbf{y}(k; t))^{-1},$$

where $\alpha_t = \{z_t \rightarrow z_t y_t, x_i \rightarrow x_i, x_i \neq z_t\}$.

It follows immediately that $\mu(\mathbf{z}(k) \rightarrow \mathbf{y}(k))$ is an automorphism of F which has the required properties.

As a corollary to Lemma 2 we deduce,

Lemma 3. Let $u = u'(\hat{x}_1) \prod_{\mathbf{y}} [x_1, y_1, \dots, y_{c-1}]^{a(\mathbf{y})}$, $c \geq 2$, be an element of $\gamma_c(F)$ where the component $u'(\hat{x}_1)$ does not involve the generator x_1 and $\mathbf{y} = (y_2, \dots, y_{c-1})$ with $y_i \in \{x_1, x_2, \dots, x_n\}$ ($y_1 \neq x_1$), $a(\mathbf{y}) \in \mathbb{Z}$. Then there is an IA-automorphism μ of F such that $\mu(x_1) \equiv x_1 u \pmod{\gamma_{c+1}(F)}$, $\mu(x_i) \equiv x_i \pmod{F'}$ for $2 \leq i \leq c$, and $\mu(x_j) = x_j$ for $j \geq c+1$.

Proof. Since $u'(\hat{x}_1)$ does not involve the generator x_1 , there is an automorphism α of F of the form given by $\alpha = \{x_1 \rightarrow x_1 u'(\hat{x}_1), x_i \rightarrow x_i, i \neq 1\}$.

Thus applying α^{-1} to u , we may assume $u = \prod_y [x_1, y_1, \dots, y_{c-1}]^{a(y)}$. By Lemma 2, for each $[x_1, y_1, \dots, y_{c-1}]$ there is an IA-automorphism $\mu(\mathbf{z} \rightarrow \mathbf{y}) \in \text{Aut}(F)$ such that $\mu(\mathbf{z} \rightarrow \mathbf{y})(x_1) \equiv x_1 [x_1, y_1, \dots, y_c] \pmod{\gamma_{c+1}(F)}$, $\mu(\mathbf{z} \rightarrow \mathbf{y})(x_i) \equiv x_i \pmod{F'}$ for $2 \leq i \leq c$, and $\mu(\mathbf{z} \rightarrow \mathbf{y})(x_j) = x_j$ for $j \geq c+1$. Define $\mu = \prod_y (\mu(\mathbf{z} \rightarrow \mathbf{y}))^{a(y)}$. It is easily verified that μ has the required properties.

LIFTING PRIMITIVITY OF SYSTEMS OF ELEMENT

We first establish the lifting of a single primitive element. We first prove,

Theorem A. *Let $n \geq c \geq 2$ and $u \in F'$ be fixed. Then there is an IA-automorphism δ of F such that $\delta(x_1 u) \equiv x_1 \pmod{\gamma_{c+1}(F)}$, $\delta(x_i) \equiv x_i \pmod{F'}$ for $i = 2, \dots, c$, and $\delta(x_j) = x_j$ for $j = c+1, \dots, n$.*

Proof. Repeated applications of Lemma 3 yield IA-automorphisms $\delta_2, \delta_3, \dots, \delta_c$ of F satisfying,

$$\begin{aligned} \delta_2(x_1 u) &= x_1 u_3, \quad u_3 \in \gamma_3(F) \quad \text{and} \quad \delta_2(x_j) = x_j \quad \text{for } j \geq 3. \\ \delta_3(x_1 u_3) &= x_1 u_4, \quad u_4 \in \gamma_4(F) \quad \text{and} \quad \delta_3(x_j) = x_j \quad \text{for } j \geq 4; \\ \delta_c(x_1 u_c) &= x_1 u_{c+1}, \quad u_{c+1} \in \gamma'_{c+1}(F) \quad \text{and} \quad \delta_c(x_j) = x_j = x_j \quad \text{for } j \geq c+1. \end{aligned}$$

Thus $\delta = \delta_c \cdots \delta_3 \delta_2$ is such that $\delta(x_1 u) \equiv x_1 \pmod{\gamma_{c+1}(F)}$, $\delta(x_i) \equiv x_i \pmod{F'}$ for $i = 2, \dots, c$, and $\delta(x_j) = x_j$ for $j \geq c+1$, as required.

We can now prove primitive lifting of a single element.

Theorem B. *Let $n \geq c \geq 2$ be fixed and let $w \in F$ be primitive mod $\gamma_{c+1}(F)$. Then w can be lifted to a primitive element of F .*

Proof. Let $w \in F$ be primitive mod $\gamma_{c+1}(F)$. For any automorphism ξ of F , the set $\{\xi(x_1), \xi(x_2), \dots, \xi(x_n)\}$ forms a basis of F and it suffices, therefore, to prove that there exists $\xi \in \text{Aut}(F)$ with $\xi(x_1) = wv$ for some v in $\gamma_{c+1}(F)$. Since w is primitive mod F' , there exists an automorphism $\alpha \in \text{Aut}(F)$ such that $\alpha(w) = x_1 u$ for some u in F' . By Theorem A, there exists an automorphism δ of F such that $\delta^{-1}(x_1) \equiv x_1 u \pmod{\gamma_{c+1}(F)}$, so that $\alpha^{-1} \delta^{-1}(x_1) \equiv w \pmod{\gamma_{c+1}(F)}$. Thus $\xi = \alpha^{-1} \delta^{-1}$ is an automorphism of F with $\xi(x_1) = wv$ for some v in γ_{c+1} . This completes the proof of the theorem.

Let $\mathbf{w} = (w_1, \dots, w_m)$, $m \geq 1$, be an arbitrary system of primitive elements mod $\gamma_{c+1}(F)$. We can now prove our principal result of the paper.

Theorem C. *For $n \geq c + m - 1$, every primitive system*

$$\mathbf{w} = (w_1, \dots, w_m) \pmod{\gamma_{c+1}(F)}$$

can be lifted to a primitive system of F .

Proof. As before we may assume, without loss of generality, that $\mathbf{w}$ is of the form $\mathbf{w} = (x_1 u_1, \dots, x_m u_m)$, $u_i \in F'$. So the problem reduces to showing that there exist $v_i \in \gamma_{c+1}(F)$, $i = 1, \dots, m$, such that

$$\mathbf{w}^* = (x_1 u_1 v_1, \dots, x_m u_m v_m)$$

is a primitive system of F . We achieve this by proving, by induction on $m \geq 1$, that for the given system $\mathbf{w} = (x_1 u_1, \dots, x_m u_m)$ there is an IA-automorphism $\sigma(\mathbf{w})$ of F such that $\sigma(\mathbf{w})(x_i) \equiv x_i u_i \pmod{\gamma_{c+1}(F)}$ for $i =$

$1, \dots, m$. For $m = 1$, the result comes from Theorem A. Let $m \geq 2$ and assume the result for $m - 1$. By the induction hypothesis, the subsystem $(x_2u_2, \dots, x_mu_m)$ lifts to a primitive subsystem $(x_2u_2v_2, \dots, x_mu_mv_m)$ of F . So there is an IA-automorphism ξ such that $\xi(x_iu_i) \equiv x_i \pmod{\gamma_{c+1}(F)}$, $i = 2, \dots, m$. Applying ξ to the given system $\mathbf{w} = (x_1u_1, \dots, x_mu_m)$ shows that $\xi(\mathbf{w}) = (x_1u_1^*, x_2v_2, \dots, x_mv_m)$ where $u_1^* \in F'$, $v_i \in \gamma_{c+1}(F)$. Thus it suffices to exhibit an automorphism δ of F with the property that $\delta(x_1u_1^*) \equiv x_1 \pmod{\gamma_{c+1}(F)}$ and $\delta(x_i) = x_i$ for $i = 2, \dots, m$. Indeed, since $n \geq c + m - 1$, Theorem A yields an automorphism δ such that $\delta(x_1u_1^*) \equiv x_1 \pmod{\gamma_{c+1}(F)}$, $\delta(x_i) \equiv x_i \pmod{F'}$ for $i = m + 1, \dots, c + m - 1$, and $\delta(x_j) = x_j$ for $j = 2, \dots, m, c + m, \dots, n$. This completes the proof of the theorem.

CONCLUDING REMARKS

For $n = 2$, $c \geq 3$, there is a primitive element $\text{mod } \gamma_{c+1}(F)$ which cannot be lifted to a primitive element of F , and for $n = 3$, $c \geq 3$, there is a primitive system of two elements $\text{mod } \gamma_{c+1}(F)$ which cannot be lifted to a primitive system of F (see [5, Remarks]). For $c \geq 4$, $n = c - 1$, it would be of interest to know whether every primitive element $\text{mod } \gamma_{c+1}(F)$ can be lifted to a primitive element of F . The simplest case of the problem is to decide whether or not, for $n = 3$, $c = 4$, the element $x_1[x_1, x_2, x_2, x_3]$ can be lifted to a basis of F .

Added in proof. A weaker form of Theorem C is also implicit in Bryant and Macedonska [J. Algebra **121** (1989), 388–398] where it is shown that for sufficiently large $n = n(m, c)$ every primitive system $\text{mod } \gamma_{c+1}(F_n)$ can be lifted to a primitive system of F_n .

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