A REMARK ON THE CONSTANTS OF THE LITTLEWOOD-PALEY INEQUALITY

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(Communicated by J. Marshall Ash)

Dedicated to V. Xanthopoulos

ABSTRACT. If $f \in H^p$, $p > 1$, and γ denotes its Littlewood-Paley square function, then $\|\gamma\|_p \leq B_p \|f\|_p$ with $B_p = O((p-1)^{-1})$, $p \rightarrow 1^+$.

1. INTRODUCTION

Let f be a 2π -periodic function in L^p , $1 < p < \infty$, $0 \leq n_0 < n_1 < \dots$ a lacunary sequence of integers, $n_{j+1}/n_j \geq \lambda > 1$, $j = 1, 2, \dots$, and

$$(1) \quad \gamma(x) = \left\{ \sum_{j=0}^{\infty} |S_{n_j}(x) - S_{n_{j-1}}(x)|^2 \right\}^{1/2},$$

where S_n , $n = 0, 1, \dots$, denote the (symmetric) partial sums of the Fourier series of f and $S_{n_{-1}} \equiv 0$. A well-known theorem of Littlewood and Paley asserts that

$$(2) \quad A_p \|f\|_p \leq \|\gamma\|_p \leq B_p \|f\|_p,$$

where the positive constants A_p and B_p depend, for fixed λ , on p only. Recently Bourgain [B] determined the asymptotic behavior of B_p by showing

$$(3) \quad B_p = O(p), \quad p \rightarrow \infty, \quad B_p = O((p-1)^{-3/2}), \quad p \rightarrow 1^+.$$

In this note we show that this "asymmetry" disappears if we restrict ourselves to H^p functions. More precisely we prove

Theorem. If $f \in H^p$, $p > 1$, then (2) is valid with $B_p = O((p-1)^{-1})$, $p \rightarrow 1^+$.

It is easy to see that this estimate, as well Bourgain's estimate $B_p = O(p)$, $p \rightarrow \infty$, cannot be improved. To this end, take, for instance, $\lambda = 2$, $n_1 = N$, $n_2 = 2N$, with N sufficiently large, and consider the functions $e^{iNx} f(x)$, $e^{iNx} g(x)$, where f, g are real trigonometric polynomials of degree N whose conjugates $\tilde{f}, \tilde{g}$ satisfy the inequalities $\|\tilde{f}\|_p > C/(p-1)$ and $\|\tilde{g}\|_p > Cp$, $1 < p < \infty$. The existence of such polynomials follows easily from the estimates

Received by the editors October 15, 1990.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 42A50, 30D55.

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0002-9939/92 \$1.00 + \$.25 per page

in [Z, Chapter VII, §3]. Here and in what follows *the letter C will denote an absolute positive constant*, not necessarily the same in all its occurrences. The proof which follows has its origin in the proof of the classical theorem of Paley on (H^1, L^2) multipliers [Z, Chapter XII, Theorem 7.8].

2. PROOF OF THE THEOREM

To facilitate the exposition we assume that $\lambda > 2$. In the opposite case standard modifications of the argument lead again to the desired result at the cost of a factor which depends on λ only. We may also assume that f is a (power type) polynomial and $p < 2$.

For any $g \in H^p$ we write $\sigma_j(g, x)$, $j \geq 0$, for the (de La Vallée-Poussin type) means $\sum_{k=0}^{\infty} \hat{g}(k)\theta(k)e^{ikx}$, where the continuous function θ is 0 in $[n_{j+1}, \infty]$, 1 in $[0, n_j]$, and linear in $[n_j, n_{j+1}]$. We write also $\Delta_j(\equiv \Delta_j(g, x))$ for $\sigma_j - \sigma_{j-3}$, $j \geq 3$.

We shall make use of the following facts:

- (a) $|\sigma_j(g, x)| \leq CM_g(x)$, $j \geq 0$, where M_g the maximal function of g .
- (b) $f = f_1 f_2$, where $f_1, f_2 \in H^{2p}$, and $|f_1|^2 = |f_2|^2 = |f|$.
- (c) The vector valued analogue of Riesz's theorem (boundedness in L^p of the partial sums of Fourier series) with constant $O((p-1)^{-1})$, $p \rightarrow 1^+$.
- (d) The Littlewood-Paley inequality (2) in the form $\|\delta(g)\|_p \leq C\|g\|_p$ for $2 < p < 4$, where $\delta(g) = \{\sum_{j=3}^{\infty} |\Delta_j(g)|^2\}^{1/2}$ (reminder: For (a)–(c), see, e.g., [Z, Chapter IV, Theorem 7.8; Chapter VII, §7; Chapter XV, Theorem 2.15], respectively).

All these facts are well known. (d) is the basic lemma in several known proofs of (2) (see, e.g., [C, §2]).

We pass now to the proof of the theorem. Starting with the decompositions $f_i = \sigma_{j-3}(f_i) + \Delta_j(f_i) + (f_i - \sigma_j(f_i))$, $i = 1, 2$, and forming the product $f = f_1 f_2$ we see that only the part

$$(4) \quad E_j = \sigma_{j-3}(f_1)\Delta_j(f_2) + \sigma_{j-3}(f_2)\Delta_j(f_1), \quad j \geq 3,$$

has spectrum which intersects $[n_{j-1}, n_j]$. Indeed, the part $\sigma_{j-3}(f_1)\sigma_{j-3}(f_2)$ has its spectrum in $[0, n_{j-1})$, because of the hypothesis $\lambda > 2$, and the other parts M in (n_j, ∞) , because $f_1, f_2 \in H^{2p}$. On writing $\gamma_1 = \{\sum_{j=3}^{\infty} |E_j|^2\}^{1/2}$ and using Riesz's theorem and (c) we obtain

$$(5) \quad \begin{aligned} \|\gamma\|_p &\leq \sum_{j=0}^2 \|S_{n_j}(f) - S_{n_{j-1}}(f)\|_p + C_p \|\gamma_1\|_p \\ &\leq C_p (5\|f\|_p + \|\gamma_1\|_p), \end{aligned}$$

with $C_p = O((p-1)^{-1})$, $p \rightarrow 1^+$. The definition of E_j and (a) give

$$(6) \quad \gamma_1 \leq 2M_{f_1}\delta(f_2) + 2M_{f_2}\delta(f_1)$$

and hence, using (d), (b), and the Hardy-Littlewood maximal function theorem [Z, Chapter I, §13],

$$(7) \quad \begin{aligned} \|\gamma_1\|_p &\leq 2\|M_{f_1}\|_{2p}\|\delta(f_2)\|_{2p} + 2\|M_{f_2}\|_{2p}\|\delta(f_1)\|_{2p} \\ &\leq C\|f_1\|_{2p}\|f_2\|_{2p} = C\|f\|_p. \end{aligned}$$

Combining (5) and (7) we obtain $\|\gamma\|_p \leq B_p \|f\|_p$ with $B_p = O((p-1)^{-1})$, $p \rightarrow 1^+$, and the proof is complete.

3. SOME REMARKS

Remarks. (i) The above argument, for fixed $p > 1$, gives $O((\lambda-1)^{-2})$ for B_p (the constants C in (a) and (d) are replaced by $C(\lambda-1)^{-1}$), while examples (e.g., variants of the one used in [B] for the $(p-1)^{-3/2}$ result) show that it cannot be better than $O((\lambda-1)^{-3/2})$.

(ii) It is not clear whether positive results for A_p or B_p , $p \rightarrow \infty$, which appear to be related to good estimates for the constant in (d), can be obtained along the lines of the present proof. The example of lacunary f 's and the content of [B] seem to hint toward the estimate $O(p^{+1/2})$, $p \rightarrow \infty$, for A_p^{-1} (such an estimate, coupled with $B_p = O(p)$, $p \rightarrow \infty$, leads easily to the $(p-1)^{-3/2}$ result). The details will appear in [P].

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