CONTINUITY OF MULTIDIMENSIONAL BROWNIAN LOCAL TIMES

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ABSTRACT. The local time of a multidimensional semimartingale at a hypersurface will be defined via Tanaka's formula. One can define a certain distance between hypersurfaces so that the continuity properties of local time can be discussed when the underlying process is Brownian motion.

1. INTRODUCTION

The local time of one-dimensional Brownian motion is very important additive functional in the construction of a general diffusion; see Itô and McKean [6] or Freedman [4]. That such construction is possible depends, besides others, on the fact that the local time can be made jointly continuous in the state and the time variables. As for the multidimensional case the local time at a state does not exist because Brownian motion never hits points. Nevertheless one can define in a natural way the local time at certain $(d-1)$ -dimensional sets when the state space has dimension d . By introducing a metric between these sets, the continuity properties can be discussed when the underlying process is Brownian motion. Bass [1] has dealt with similar problems in a more general set up. Our approach here is more direct and specific. The main tool is Tanaka's formula. We shall employ this formula to handle continuity in an explicit manner. For other related works, see Yor [12].

2. DEFINITION AND MOMENT INEQUALITIES

Recall that if $X = (X_t)$ is a real continuous semimartingale, the local time of X at a point a , denoted by $L_t^a(X)$, is defined through the Tanaka formula

$$(X_t - a)^+ = (X_0 - a)^+ + \int_0^t I_{[X_s > a]} dX_s + \frac{1}{2} L_t^a(X),$$

where $x^+ = x \vee 0$, I_A is the indicator function of the set A ; see Meyer [9]. If $X = (X^1, X^2, \dots, X^d)$ is an d -dimensional continuous semimartingale (each coordinate is a real continuous semimartingale) and if C is a class C^2 hypersurface; i.e., $C = \{x = (x_1, x_2, \dots, x_d) : x_d = g(x_1, x_2, \dots, x_{d-1})\}$,

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$g \in C^2(R^{d-1})$, then we define the *local time of X at C* to be $L_t^0(Y)$, where $Y = X^d - g(X^1, X^2, \dots, X^{d-1})$. Let us remark here that the definition is free of ambiguity and able to be extended to more general $(d-1)$ -dimensional sets by localization. For $g \in C^2(R^{d-1})$, let $g_i = \partial g / \partial x_i$, $g_{ii} = \partial^2 g / \partial x_i^2$, where $i = 1, 2, \dots, d-1$; for $R > 0$, let $\|g\|_R = \sup\{|g(x)| + \sum_{i=1}^{d-1} |g_i(x)| + \sum_{i=1}^{d-1} |g_{ii}(x)| : |x| \leq R\}$, $|\cdot|$ is the usual norm of R^{d-1} . Let $B_s = (B_s^1, B_s^2, \dots, B_s^d)$, $s \geq 0$, be d -dimensional Brownian motion. To discuss the continuity properties, we establish first the following moment inequality.

Theorem 2.1. Fix $T, R, m > 0$, and let $\tau(R) = \inf\{s \geq 0 : \sum_{i=1}^{d-1} (B_s^i)^2 \geq R\}$. If $f, g \in C^2(R^{d-1})$, $\|f - g\|_R = \delta \leq 1$, $\|f\|_R \leq m$, $\|g\|_R \leq m$, then for any $x \in R^d$ we have

$$(2.1) \quad E^x(L(f, \tau(R) \wedge T) - L(g, \tau(R) \wedge T))^{2n} \leq C(n, m, T)\delta^n,$$

where

$$L(f, t) = L_t^0(B^d - f(B^1, \dots, B^{d-1})), \quad L(g, t) = L_t^0(B^d - g(B^1, \dots, B^{d-1})),$$

and $C(n, m, T)$ is a constant depending on n, m, T .

Proof. According to the definition, let us write

$$\begin{aligned} L(f, \tau(R) \wedge T) &= (B_{\tau(R) \wedge T}^d - f(B_{\tau(R) \wedge T}^1, \dots, B_{\tau(R) \wedge T}^{d-1}))^+ \\ &\quad - (B_0^d - f(B_0^1, \dots, B_0^{d-1}))^+ \\ &\quad - \int_0^{\tau(R) \wedge T} I_{[B_s^d > f(B_s^1, \dots, B_s^{d-1})]} dB_s^d \\ &\quad + \int_0^{\tau(R) \wedge T} I_{[B_s^d > f(B_s^1, \dots, B_s^{d-1})]} df(B_s^1, \dots, B_s^{d-1}). \end{aligned}$$

Applying the Itô formula to the fourth term, we get

$$\begin{aligned} L(f, \tau(R) \wedge T) &= (B_{\tau(R) \wedge T}^d - f(B_{\tau(R) \wedge T}^1, \dots, B_{\tau(R) \wedge T}^{d-1}))^+ \\ &\quad - (B_0^d - f(B_0^1, \dots, B_0^{d-1}))^+ - \int_0^{\tau(R) \wedge T} I_{[B_s^d > f(B_s^1, \dots, B_s^{d-1})]} dB_s^d \\ &\quad + \sum_{i=1}^{d-1} \int_0^{\tau(R) \wedge T} f_i(B_s^1, \dots, B_s^{d-1}) I_{[B_s^d > f(B_s^1, \dots, B_s^{d-1})]} dB_s^i \\ &\quad + \frac{1}{2} \sum_{i=1}^{d-1} \int_0^{\tau(R) \wedge T} f_{ii}(B_s^1, \dots, B_s^{d-1}) I_{[B_s^d > f(B_s^1, \dots, B_s^{d-1})]} ds. \end{aligned}$$

Expressing $L(g, \tau(R) \wedge T)$ in the same way and then taking the absolute dif-

ference, we get

$$\begin{aligned}
& |L(f, \tau(R) \wedge T) - L(g, \tau(R) \wedge T)| \\
& \leq |f(B_{\tau(R) \wedge T}^1, \dots, B_{\tau(R) \wedge T}^{d-1}) - g(B_{\tau(R) \wedge T}^1, \dots, B_{\tau(R) \wedge T}^{d-1})| \\
& \quad + |f(B_0^1, \dots, B_0^{d-1}) - g(B_0^1, \dots, B_0^{d-1})| \\
& \quad + \sum_{i=1}^{d-1} \left| \int_0^{\tau(R) \wedge T} \{f_i(B_s^1, \dots, B_s^{d-1}) I_{[B_s^d > f(B_s^1, \dots, B_s^{d-1})]} \right. \\
& \quad \quad \left. - g_i(B_s^1, \dots, B_s^{d-1}) I_{[B_s^d > g(B_s^1, \dots, B_s^{d-1})]} \} dB_s^i \right| \\
& \quad + \frac{1}{2} \sum_{i=1}^{d-1} \left| \int_0^{\tau(R) \wedge T} \{f_{ii}(B_s^1, \dots, B_s^{d-1}) I_{[B_s^d > f(B_s^1, \dots, B_s^{d-1})]} \right. \\
& \quad \quad \left. - g_{ii}(B_s^1, \dots, B_s^{d-1}) I_{[B_s^d > g(B_s^1, \dots, B_s^{d-1})]} \} ds \right|.
\end{aligned}$$

Without loss of generality, we may discuss only the case $d = 2$ and assume $|x| < R$. Of course,

$$E^x(f(B_{\tau(R) \wedge T}^1) - g(B_{\tau(R) \wedge T}^1))^{2n} \leq \delta^{2n} \leq \delta^n$$

and

$$E^x(f(B_0^1) - g(B_0^1))^{2n} \leq \delta^{2n} \leq \delta^n.$$

Observe that

$$\begin{aligned}
& E^x \int_0^{\tau(R) \wedge T} |I_{[B_s^2 > f(B_s^1)]} - I_{[B_s^2 > g(B_s^1)]}| ds \\
& \leq \int_0^T \int_{-R}^R \int_{f(u) \wedge g(u)}^{f(u) \vee g(u)} \frac{1}{2\pi s} \exp \left[-\frac{(u-x_1)^2 + (v-x_2)^2}{2s} \right] dv du ds, \quad x = (x_1, x_2), \\
& \leq \int_0^T \left(\int_{-R}^R \frac{\delta}{\sqrt{2\pi s}} \exp \left[-\frac{(u-x_1)^2}{2s} \right] du \right) \frac{1}{\sqrt{2\pi s}} ds \leq C(T)\delta.
\end{aligned}$$

So the Markov property will ensure

$$\begin{aligned}
& E^x \left(\int_0^{\tau(R) \wedge T} |I_{[B_s^2 > f(B_s^1)]} - I_{[B_s^2 > g(B_s^1)]}| ds \right)^n \\
& \leq E^x \left(\int_0^T |I_{[B_{s \wedge \tau(R)}^2 > f(B_{s \wedge \tau(R)}^1)]} - I_{[B_{s \wedge \tau(R)}^2 > g(B_{s \wedge \tau(R)}^1)]}| ds \right)^n \\
& \leq n! E^x \int_0^T \int_{s_1}^T \cdots \int_{s_{n-1}}^T \prod_{i=1}^n |I_{[B_{s_i \wedge \tau(R)}^2 > f(B_{s_i \wedge \tau(R)}^1)]} - I_{[B_{s_i \wedge \tau(R)}^2 > g(B_{s_i \wedge \tau(R)}^1)]}| ds_i \\
& \leq n! E^x \int_0^T E^{(B_{s_1}^1, B_{s_1}^2)} \int_{s_1}^T \cdots E^{(B_{s_{n-1}}^1, B_{s_{n-1}}^2)} \int_{s_{n-1}}^T \\
& \quad \times \prod_{i=1}^n |I_{[B_{s_i \wedge \tau(R)}^2 > f(B_{s_i \wedge \tau(R)}^1)]} - I_{[B_{s_i \wedge \tau(R)}^2 > g(B_{s_i \wedge \tau(R)}^1)]}| ds_i \\
& \leq C(n, T)\delta^n.
\end{aligned}$$

By the Burkholder inequality, we see

$$\begin{aligned} & E^x \left\{ \int_0^{\tau(R) \wedge T} |I_{[B_s^2 > f(B_s^1)]} - I_{[B_s^2 > g(B_s^1)]}| dB_s^2 \right\}^{2n} \\ & \leq C(n) E^x \left\{ \int_0^{\tau(R) \wedge T} |I_{[B_s^2 > f(B_s^1)]} - I_{[B_s^2 > g(B_s^1)]}| ds \right\}^n \\ & \leq C(n, T) \delta^n \end{aligned}$$

and

$$\begin{aligned} & E^x \left\{ \int_0^{\tau(R) \wedge T} ((f_1(B_s^1)I_{[B_s^2 > f(B_s^1)]} - g_1(B_s^1)I_{[B_s^2 > g(B_s^1)]}) dB_s^1 \right\}^{2n} \\ & \leq C(n) E^x \left\{ \int_0^{\tau(R) \wedge T} ((f_1(B_s^1)I_{[B_s^2 > f(B_s^1)]} - g_1(B_s^1)I_{[B_s^2 > g(B_s^1)]})^2 ds \right\}^n \\ & \leq C(n) E^x \left\{ \int_0^{\tau(R) \wedge T} (f_1(B_s^1) - g_1(B_s^1))^2 ds \right\}^n \\ & \quad + C(n) E^x \left\{ \int_0^{\tau(R) \wedge T} g_1^2(B_s^1) |I_{[B_s^2 > f(B_s^1)]} - I_{[B_s^2 > g(B_s^1)]}| ds \right\}^n \\ & \leq C(n, T) \delta^{2n} + C(n, m, T) \delta^n \\ & \leq C(n, m, T) \delta^n. \end{aligned}$$

By the same token, we have

$$\begin{aligned} & E^x \left\{ \int_0^{\tau(R) \wedge T} |f_{11}(B_s^1)I_{[B_s^2 > f(B_s^1)]} - g_{11}(B_s^1)I_{[B_s^2 > g(B_s^1)]}| ds \right\}^{2n} \\ & \leq C(n) E^x \left\{ \int_0^{\tau(R) \wedge T} |f_{11}(B_s^1) - g_{11}(B_s^1)| ds \right\}^{2n} \\ & \quad + C(n) E^x \left\{ \int_0^{\tau(R) \wedge T} |g_{11}(B_s^1)(I_{[B_s^2 > f(B_s^1)]} - I_{[B_s^2 > g(B_s^1)]})| ds \right\}^{2n} \\ & \leq C(n, T) \delta^{2n} + C(n, m, T) \delta^{2n} \\ & \leq C(n, m, T) \delta^n. \end{aligned}$$

The conclusion follows by lumping the above inequalities together. $\square$

Corollary 2.2. *Under the assumptions of Theorem 2.1, if $0 \leq s, t \leq T$ then for any $x \in R^d$ we have*

$$E^x(L(f, \tau(R) \wedge s) - L(g, \tau(R) \wedge t))^{2n} \leq C(n, m, T)(\delta^n + |s - t|^n).$$

Proof. Inspecting the proof in Theorem 2.1, we see

$$E^x \sup_{0 \leq t \leq T} (L(f, \tau(R) \wedge s) - L(g, \tau(R) \wedge t))^{2n} \leq C(n, m, T)|s - t|^n.$$

Hence

$$E^x(L(f, \tau(R) \wedge s) - L(g, \tau(R) \wedge t))^{2n} \leq C(n, m, T)(\delta^n + |s - t|^n). \quad \square$$

By a well-known theorem of Kolmogorov, Corollary 2.2 implies

Corollary 2.3. $L(f, t)$ can be modified so that it is jointly continuous in f and t whenever f is restricted in a finite-dimensional subspace of $C^2(R^{d-1})$.

Let H be the set of hyperplanes in R^d . Then H can be considered a finite-dimensional subset of $C^2(R^{d-1})$. Therefore we deduce the following result due to Bass [2].

Corollary 2.4. $L(f, t)$ considered as a random field on $H \times [0, \infty)$, can have a continuous version.

3. MODULUS OF CONTINUITY

Since a hyperplane can be expressed as a set of the form $\{x: cx = l\}$, where $x = (x_1, x_2, \dots, x_d) \in R^d$, $c = (c_1, \dots, c_d) \in R^d$, $|c| = 1$, $l \in R^1$, we will write $L(c, l, t)$ instead of $L(f, t)$ whenever dealing with the continuity problem of $L(f, t)$ restricted in $H \times [0, \infty)$. From Corollary 2.4, we know $L(c, l, t)$ is jointly continuous in c, l and t . Furthermore, Theorem 2.1 implies for every α , $0 < \alpha < \frac{1}{2}$, we have

$$|L(c, l, t) - L(c', l', t')| \leq C(|c - c'| + |l - l'| + |t - t'|)^\alpha$$

by a more refined version of Kolomogorov's theorem. In the following stronger results will be obtained.

Theorem 3.1. For each c we have almost surely

$$(3.1) \quad \overline{\lim}_{\delta \downarrow 0} \sup_l \frac{L(c, l + \delta, t) - L(c, l, t)}{(\delta \log \delta^{-1})^{1/2}} = 2[\sup_l L(c, l, t)]^{1/2}$$

and

$$(3.2) \quad \overline{\lim}_{\substack{0 \leq s \leq t \leq 1 \\ t-s=\delta \downarrow 0}} \sup_l \frac{L(c, l, t) - L(c, l, s)}{(2\delta \log \delta^{-1})^{1/2}} = 1.$$

Proof. (3.1) follows from results of McKean [8] and Ray [10]. (3.2) is a result of Csaki, et al. [3]. $\square$

The problem of getting a modulus of continuity with respect to c is more difficult. We present here a result in several steps. Let $S = \{u \in R^d: |u| = 1\}$. Call a finite set $U(\varepsilon)$ and ε -net of S , if $U(\varepsilon) \subseteq S$, $|x - U(\varepsilon)| = \inf\{|x - y|: y \in U(\varepsilon)\} \leq \varepsilon$ for every $x \in S$, and $|x - y| \geq \varepsilon$ for every $x, y \in U(\varepsilon)$. Obviously, the number of points in $U(\varepsilon)$ is bounded by $C(1/\varepsilon)^{d-1}$ for a constant C independent of ε and there exists a sequence $U(1) \subseteq U(2) \subseteq \dots \subseteq U(n) \subseteq \dots$ such that $U(n)$ is an 2^{-n} -net of S . Let $I = I_{[0, \infty)}$, $E = E^x$, $P = P^x$, $uB_s = \sum_{i=1}^d u_i B_s^i$, $u = (u_1, \dots, u_d) \in R^d$.

Lemma 3.2. For each $t \geq 0$ we have almost surely

$$(3.3) \quad \overline{\lim}_{\substack{u, v \in S \\ |u-v| \downarrow 0}} \frac{\int_0^t |I(uB_s) - I(vB_s)| ds}{|u - v|(\log |u - v|^{-1})} < \infty.$$

Proof. From the proof of Theorem 2.1, we see

$$E \left(\int_0^{t \wedge \tau(R)} |I(uB_s) - I(vB_s)| ds \right)^m \leq m! C(t)^m |u - v|^m.$$

Hence there exists a constant $c = c(t) > 0$ such that

$$E \left\{ \exp \left[\frac{1}{c|u-v|} \int_0^{t \wedge \tau(R)} |I(uB_s) - I(vB_s)| ds \right] \right\} < \infty.$$

Thus for $b > d - 1$

$$\begin{aligned} P \left(\max_{\substack{u \in U(n) \\ v \in U(n+1) \\ |u-v| \leq 2^{-n}}} \int_0^{t \wedge \tau(R)} |I(uB_s) - I(vB_s)| ds \geq bc|u-v|(\log|u-v|^{-1}) \right) \\ \leq \sum_{\substack{v \in U(n+1) \\ |u-v| \leq 2^{-n}}} \sum_{u \in U(n)} P \left(\int_0^{t \wedge \tau(R)} |I(uB_s) - I(vB_s)| ds \geq bc|u-v|(\log|u-v|^{-1}) \right) \\ \leq C2^{n(d-1)}e^{-bn}. \end{aligned}$$

The Borel-Cantelli lemma tells us

$$\overline{\lim}_{n \rightarrow \infty} \max_{\substack{u \in U(n) \\ v \in U(n+1) \\ |u-v| \leq 2^{-n}}} \int_0^{t \wedge \tau(R)} |I(uB_s) - I(vB_s)| ds / |u-v|(\log|u-v|^{-1}) < \infty, \quad \text{a.s.}$$

Employing a well-known technique (see Sheu [11] or Knight [7, p. 26]) then letting $R \uparrow \infty$, we can obtain (3.3) from the above inequality.

Lemma 3.3. *We have almost surely*

$$(3.4) \quad \overline{\lim}_{\substack{u, v \in S \\ |u-v| \downarrow 0}} \left| \int_0^t I(vB_s) d(vB_s) - \int_0^t I(uB_s) d(uB_s) \right| / |u-v|^{1/2}(\log|u-v|^{-1}) < \infty.$$

Proof. Remember that for a continuous square integrable martingale (X_t) with quadratic variational process $(\langle X \rangle_t)$, that is $(X_t^2 - \langle X \rangle_t)$ is a martingale, $(\exp(X_t - \frac{1}{2}\langle X \rangle_t))$ is a supermartingale. In particular, $E[\exp(X_t - \frac{1}{2}\langle X \rangle_t)] \leq 1$ for all t , if $X_0 = 0$. See, for example, Ikeda and Watanabe [5, p.142]. Since $(\int_0^t I(uB_s) d(uB_s) - I(vB_s) d(vB_s))$ is a martingale starting at the origin and its quadratic variational process is $(\int_0^t \sum_{i=1}^d [u_i I(uB_s) - v_i I(vB_s)]^2 ds)$, we have for any $\alpha, \beta > 0, \alpha\beta > d - 1$,

$$\begin{aligned} P \left(\int_0^t [I(uB_s) d(uB_s) - I(vB_s) d(vB_s)] \geq \frac{\alpha}{2} \int_0^t \sum_{i=1}^d [u_i I(uB_s) - v_i I(vB_s)]^2 ds + \beta \right) \\ = P \left(\alpha \int_0^t [I(uB_s) d(uB_s) - I(vB_s) d(vB_s)] \right. \\ \left. - \frac{\alpha^2}{2} \int_0^t \sum_{i=1}^d [u_i I(uB_s) - v_i I(vB_s)]^2 ds \leq \alpha\beta \right) \leq e^{-\alpha\beta}. \end{aligned}$$

Since the same bound applies to $-\int_0^t [I(uB_s)d(uB_s) - I(vB_s)d(vB_s)]$ as well, we have

$$\begin{aligned} P \left(\left| \int_0^t [I(uB_s)d(uB_s) - I(vB_s)d(vB_s)] \right| \right. \\ \left. \geq \frac{\alpha}{2} \int_0^t \sum_{i=1}^d [u_i I(uB_s) - v_i I(vB_s)]^2 ds + \beta \right) \\ \leq 2e^{-\alpha\beta}. \end{aligned}$$

Now

$$\int_0^t \sum_{i=1}^d [u_i I(uB_s) - v_i I(vB_s)]^2 ds \leq \int_0^t |I(uB_s) - I(vB_s)| ds + t|u - v|^2.$$

Therefore

$$\begin{aligned} P \left(\left| \int_0^t [I(uB_s)d(uB_s) - I(vB_s)d(vB_s)] \right| \right. \\ \left. \geq \frac{\alpha}{2} \int_0^t |I(uB_s) - I(vB_s)| ds + \frac{\alpha t}{2} |u - v|^2 + \beta \right) \\ \leq 2e^{-\alpha\beta}. \end{aligned}$$

Replacing α and β by $\alpha/\sqrt{|u - v|}$ and $\beta\sqrt{|u - v|}(\log|u - v|^{-1})$ respectively, we get

$$\begin{aligned} P \left(\frac{\left| \int_0^t [I(uB_s)d(uB_s) - I(vB_s)d(vB_s)] \right|}{\sqrt{|u - v|}(\log|u - v|^{-1})} \right. \\ \left. \geq \frac{\alpha}{2} \frac{\int_0^t |I(uB_s) - I(vB_s)| ds}{\sqrt{|u - v|}(\log|u - v|^{-1})} + \frac{t}{2} \frac{|u - v|}{(\log|u - v|^{-1})} + \beta \right) \\ \leq 2|u - v|^{\alpha\beta}. \end{aligned}$$

When n is large and $|u - v| \leq 2^{-n}$, we have $(\frac{t}{2})(|u - v|/\log|u - v|^{-1}) \leq 1$.

Thus

$$\begin{aligned}
 & P \left(\begin{aligned} & \text{Max}_{\substack{u \in U(n) \\ v \in U(n+1) \\ |u-v| \leq 2^{-n}}} \frac{\int_0^t |I(uB_s)d(uB_s) - I(vB_s)d(vB_s)|}{\sqrt{|u-v|}(\log|u-v|^{-1})} \\ & \geq \text{Max}_{\substack{u \in U(n) \\ v \in U(n+1) \\ |u-v| \leq 2^{-n}}} \frac{\alpha \int_0^t |I(uB_s) - I(vB_s)|ds}{2 |u-v|(\log|u-v|^{-1})} + \beta + 1 \end{aligned} \right) \\
 & \leq P \left(\begin{aligned} & \text{Max}_{\substack{u \in U(n) \\ v \in U(n+1) \\ |u-v| \leq 2^{-n}}} \left\{ \frac{\int_0^t |I(uB_s)d(uB_s) - I(vB_s)d(vB_s)|}{\sqrt{|u-v|}(\log|u-v|^{-1})} \right. \\ & \quad \left. - \frac{\alpha \int_0^t |I(uB_s) - I(vB_s)|ds}{2 |u-v|(\log|u-v|^{-1})} \right\} \geq \frac{t}{2} \frac{|u-v|}{\log|u-v|^{-1}} + \beta \end{aligned} \right) \\
 & \leq C 2^{-(\alpha\beta-d+1)n}.
 \end{aligned}$$

By Lemma 3.2 and the Borel-Cantelli lemma, we conclude

$$\lim_{n \rightarrow \infty} \text{Max}_{\substack{u \in U(n) \\ v \in U(n+1) \\ |u-v| \leq 2^{-n}}} \frac{|\int_0^t [I(uB_s)d(uB_s) - I(vB_s)d(vB_s)]|}{\sqrt{|u-v|}(\log|u-v|^{-1})} < \infty. \quad \text{a.s.}$$

Now the proof is completed by the same reason as in Lemma 3.2. $\square$

Theorem 3.4. For each l, t , we have almost surely

$$\lim_{\substack{u, v \in S \\ |u-v| \downarrow 0}} \frac{|L(u, l, t) - L(v, l, t)|}{\sqrt{|u-v|}(\log|u-v|^{-1})} < \infty.$$

Proof. Consider the case $l = 0$. Then

$$\begin{aligned}
 & |L(u, 0, t) - L(v, 0, t)| \\
 & \leq |u-v|(|B_t| + |B_0|) + \left| \int_0^t [I(uB_s)d(uB_s) - I(vB_s)d(vB_s)] \right|.
 \end{aligned}$$

Applying Lemma 3.3, we immediately obtain the result. The proof for the case $l \neq 0$ is identical as for the case $l = 0$. $\square$

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