A REMARK ON SINGULAR INTEGRALS WITH COMPLEX HOMOGENEITY

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ABSTRACT. The Hilbert transform can be approximated by operators with Fourier multiplier given by $c_\gamma \operatorname{sgn}(\xi)|\xi|^{i\gamma}$. If we let $\gamma \rightarrow 0$ it is known that these operators converge to the Hilbert transform in L^2 norm. It has also been shown that as $\gamma \rightarrow 0$ these operators may diverge even for functions in L^2 . In this paper it is shown that we also will have divergence if we select any sequence $\{\gamma_n\}_{n=1}^\infty$ such that $\gamma_n \rightarrow 0$. The proof makes use of Bourgain's entropy method.

INTRODUCTION

Let f be an integrable function on the real line $\mathbb{R}$. The Hilbert transform $\tilde{f}$ is the principal value convolution of f with the singular kernel $\frac{1}{\pi t}$. At almost every point, the integral defining $\tilde{f}$ converges, and hence $\tilde{f}$ exists. The existence of $\tilde{f}$ is a central fact in the theory of real and complex analysis, and one which has been generalized in a variety of directions. In the 1950s a possible new approach to establishing the existence of this important operator was suggested. In one dimension the idea was to replace convolution with $\frac{1}{\pi t}$ by convolution with $\frac{1}{\pi t}|t|^{-i\gamma}$. The hope was that this family of operators would be easier to study because of the additional cancellation induced by the term $|t|^{-i\gamma}$, and that by letting $\gamma \rightarrow 0$, the properties of the Hilbert transform could be recovered. The first step in this program was successfully completed by Muckenhoupt [5] who showed that for each fixed γ ,

$$H_\gamma f = f \star \frac{1}{\pi} \frac{\operatorname{sgn} t}{|t|^{1+i\gamma}}$$

exists, and that $H_\gamma f \rightarrow H_0 f = \tilde{f}$ in L^2 norm. Later P. Ash [2] showed that $H_\gamma f \rightarrow H_0 f = \tilde{f}$ in L^p norm, $1 < p < \infty$. Unfortunately in 1979 it was shown that $\lim_{\gamma \rightarrow 0^+} H_\gamma f(x)$ may fail to exist a.e. In [1] this is shown by construction of a function $f \in L^p(\mathbb{R})$, for all p , $1 < p < \infty$, such that

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$\limsup_{\gamma \rightarrow 0^+} H_\gamma f(x) = \infty$ a.e. This example made use of the fact that the limit was being taken through a continuous parameter. In particular, for each fixed x , there was a set of γ 's (of measure zero) such that $H_\gamma f(x) = \infty$. This left open the question of what happens if we use only a sequence $\{\gamma_j\}_{j=1}^\infty$ such that $\gamma_j \rightarrow 0$. In this paper we show that in fact for any sequence $\{\gamma_j\}_{j=1}^\infty$ such that $\lim_{j \rightarrow \infty} \gamma_j = 0$, there exists a function $f \in L^p(\mathbb{R})$, $1 < p < \infty$, such that on a set of positive measure $\lim_{j \rightarrow \infty} H_{\gamma_j} f(x)$ fails to converge.

The operators given by convolution with $\frac{1}{\pi}(\operatorname{sgn} t/|t|^{1+i\gamma})$ have Fourier multipliers given by $c_\gamma \operatorname{sgn}(\xi)|\xi|^{i\gamma}$ where $c_\gamma = 2\Gamma(-i\gamma) \sinh(\frac{\gamma\pi}{2})$ is a constant that depends on γ and converges to a finite nonzero limit as $\gamma \rightarrow 0$. (See [2, 5].)

THE PERIODIC CASE

We will first consider the easier case of $\mathbb{R}$ replaced by $T = [0, 2\pi)$. On T we consider the operators given by the multipliers $c_\gamma \operatorname{sgn}(n)|n|^{i\gamma}$. The a.e. existence of these operators has been shown by Weiss and Zygmund [8]. These operators will be denoted by $\tilde{T}_\gamma$, thus if $\varphi(x) = \sum \hat{\varphi}(n)e^{inx}$ then $\tilde{T}_\gamma \varphi(x) = \sum \hat{\varphi}(n)c_\gamma \operatorname{sgn}(n)|n|^{i\gamma}e^{inx}$.

Theorem 1. *Let $\{\gamma_j\}_{j=1}^\infty$ be a sequence such that $\lim_{j \rightarrow \infty} \gamma_j = 0$; then there exists a function $\varphi \in L^\infty(T)$ such that $\lim_{j \rightarrow \infty} \tilde{T}_{\gamma_j} \varphi(x)$ fails to exist on a set of positive measure.*

The proof will use Bourgain's Entropy method. The form of his theorem that will be most useful here is stated below. (See [3, 6] as well as Bourgain's original paper [4] for additional applications of this useful result.)

Bourgain's Entropy Theorem [4, Proposition 2]. *Let (Ω, μ) denote a probability space. Assume $\{T_n\}_{n=1}^\infty$ is a sequence of uniformly bounded linear operators on $L^2(\Omega)$ and that these operators commute with a sequence $\{R_j\}$ of positive isometries that have $R_j(1) = 1$ and satisfy $\frac{1}{j} \sum_{j=1}^J R_j f \rightarrow \int f d\mu$ in mean for $f \in L^1(\Omega)$. Under these conditions, if there is a number $\beta > 0$ such that for each $N \geq 1$ there is a function φ (which is allowed to depend on N), $\|\varphi\|_2 \leq 1$, and a collection $\{T_{n_i}\}_{i=1}^N$ such that $\|T_{n_i}\varphi - T_{n_j}\varphi\|_2 > \beta$ for each $i \neq j$, $i = 1, \dots, N$, $j = 1, \dots, N$, then there is bounded function φ such that $\{T_n \varphi\}$ is not a.e. convergent.*

In our application we will let $\Omega = [0, 2\pi)$, μ be the normalized Lebesgue measure, $T_n = \tilde{T}_{\gamma_n}/c_{\gamma_n}$ and $R_j(x) = x + j\alpha \pmod{2\pi}$, where α can be any real number not rationally related to 2π . Since we know that $\lim_{n \rightarrow \infty} c_{\gamma_n}$ exists and is nonzero, if we can show that $\lim_{n \rightarrow \infty} T_n$ fails to exist, we can conclude that $\lim_{n \rightarrow \infty} \tilde{T}_{\gamma_n}$ fails to exist.

The required function φ will have the form

$$\varphi(x) = \frac{1}{\sqrt{L}} \sum_{n=1}^L e^{i2^n x},$$

where L will be determined later. Note that $\|\varphi\|_2 = 1$. For this function, we have

$$\tilde{T}_\gamma \varphi(x) = \frac{1}{\sqrt{L}} \sum_{n=1}^L c_\gamma (\operatorname{sgn} 2^n) e^{i\gamma \cdot n \cdot \ln 2} e^{i2^n x}.$$

Consequently,

$$T_r \varphi(x) - T_s \varphi(x) = \frac{1}{\sqrt{L}} \sum_{n=1}^L (\operatorname{sgn} 2^n) e^{i2^n x} e^{i\gamma_r \cdot n \cdot \ln 2} (1 - e^{i(\gamma_s - \gamma_r) n \cdot \ln 2}),$$

and thus

$$\begin{aligned} \|T_r \varphi - T_s \varphi\|_2^2 &= \frac{1}{L} \sum_{n=1}^L |1 - e^{i(\gamma_s - \gamma_r) n \cdot \ln 2}|^2 \\ &= \frac{1}{L} \sum_{n=1}^L 2(1 - \cos[(\gamma_s - \gamma_r) n \cdot \ln 2]). \end{aligned}$$

Let $\theta_{sr} = \frac{1}{2\pi}(\gamma_s - \gamma_r) \ln 2$, and define $\Omega = \{\theta_{sr} : 1 \leq r < s \leq N\}$. We need to find an L such that for each $\theta \in \Omega$, we have $\frac{1}{L} \sum_{n=1}^L 2(1 - \cos 2\pi n\theta) > \beta$. Let $\Omega = \Omega_1 \cup \Omega_2$ where

$$\Omega_1 = \{\theta : \theta \in \Omega, \theta \text{ irrational}\} \text{ and } \Omega_2 = \{\theta : \theta \in \Omega, \theta \text{ rational}\}.$$

There are two cases. If $\theta \in \Omega_1$, we know that $\{n\theta\}$ is uniformly distributed mod 1, and consequently, since $(1 - \cos 2\pi x)$ is Riemann integrable and $\int_0^1 2(1 - \cos 2\pi x) dx = 2$, there exists an $l = l(\theta)$ such that

$$\frac{1}{L} \sum_{n=1}^L 2(1 - \cos 2\pi n\theta) > \left(\frac{1}{2}\right) 2 = 1$$

for all $L > 1$. There are only a finite number of numbers in Ω_1 , we let $L_1 = \max_{\theta \in \Omega_1} l(\theta)$.

If $\theta \in \Omega_2$ then θ is rational; we need to prove a similar inequality. Assume $\theta = p/q$, where p and q are integers and $(p, q) = 1$. Then we have $\{n\theta\} = \{np/q\}$. If L is a multiple of q , then the points $\{np/q\}_{n=1}^L$ are uniformly distributed in the set $\{k/q\}_{k=0}^{q-1}$. Thus at least $1/4$ of the points will be a distance of at least $1/4$ away from zero. Since $1 - \cos(2\pi x) \geq 1$ for $1/4 \leq x \leq 3/4$, we have

$$\frac{1}{L} \sum_{n=1}^L 2(1 - \cos 2\pi n\theta) \geq \frac{1}{L} \left(\frac{L}{4}\right) 2(1) = \frac{1}{2}.$$

Now let L_2 be the product $q_1 q_2 \cdots q_j$ where $\{p_i/q_i : i = 1, \dots, j\}$ represent the rationals in Ω_2 . Finally let $L = L_1 L_2$. We see that conditions are satisfied for both estimates above, and hence we have $\|T_r \varphi - T_s \varphi\|_2^2 \geq 1/2$ for $r = 1, 2, \dots, N$, $s = 1, 2, \dots, N$, $r \neq s$ as desired. $\square$

Corollary. For any p , $1 < p \leq 2$, and any $B > 0$, there exists a function $\varphi = \varphi_B$ in $L^p(\mathbb{T})$ such that

$$\left| \left\{ x : \sup_n |\tilde{T}_{\gamma_n} \varphi(x)| > \lambda \right\} \right| > \frac{B}{\lambda^p} \|\varphi\|_p^p$$

for some $\lambda > 0$. In addition, φ can be taken to be a trigonometric polynomial.

Proof. By Theorem 1, we know that we do not have convergence for some $\varphi \in L^\infty(\mathbb{T})$ and hence for some $\varphi \in L^p(\mathbb{T})$. By Stein's theorem [7], this

implies the failure of a weak type (p, p) inequality for $1 < p \leq 2$. Because trigonometric polynomials are dense, if we had a weak type inequality for all trigonometric polynomials we would have it for all $L^p(\mathbf{T})$. $\square$

THE NONCOMPACT CASE

We also can state a version of Theorem 1 for the real line.

Theorem 2. *Let $\{\gamma_j\}_{j=1}^\infty$ be a sequence such that $\lim_{n \rightarrow \infty} \gamma_n = 0$, then there exists a function $f \in L^p(\mathbb{R})$, $1 < p \leq \infty$, such that $\lim_{n \rightarrow \infty} H_{\gamma_n} f(x)$ fails to exist on a set of positive measure.*

Proof. Bourgain's Entropy Theorem requires operators to act on a probability space. Thus we cannot use his method to directly give a counterexample on $\mathbb{R}$. However we will take advantage of our example on $\mathbf{T}$. See [1] where a similar idea is used. Some steps in the computation below are given in greater detail in [1].

Let φ be a function in $L^\infty(\mathbf{T})$ such that $\lim_{n \rightarrow \infty} \tilde{T}_{\gamma_n} \varphi(x)$ fails to exist on a set E , with $|E| > 0$. Such a function exists by Theorem 1. Write $\varphi(x) = \sum_{k=-\infty}^\infty a_k e^{ikx}$. We can assume without loss of generality that $\int_{-\pi}^\pi \varphi(x) dx = 0$, and hence that $a_0 = 0$. Define a function f on $\mathbb{R}$ by

$$\hat{f}(\xi) = \sum_{k=-\infty}^\infty a_k \chi_{(-k-1/2, -k+1/2)}(\xi).$$

Thus

$$(H_\gamma f)^\wedge(\xi) = \sum_{k=-\infty}^\infty a_k \chi_{(-k-1/2, -k+1/2)}(\xi) c_\gamma \operatorname{sgn}(\xi) |\xi|^{i\gamma}.$$

Taking inverse Fourier transforms, we have

$$\begin{aligned} H_\gamma f(x) &= \frac{1}{2\pi} \int_{-\infty}^\infty (H_\gamma f)^\wedge(\xi) e^{-i\xi x} d\xi \\ &= \frac{1}{2\pi} \int_{-\infty}^\infty \sum_{k=-\infty}^\infty a_k \chi_{(-k-1/2, -k+1/2)}(\xi) c_\gamma \operatorname{sgn}(\xi) |\xi|^{i\gamma} e^{-i\xi x} d\xi \\ &= \frac{1}{2\pi} \sum_{k=-\infty}^\infty a_k \int_{-k-1/2}^{-k+1/2} c_\gamma \operatorname{sgn}(\xi) |\xi|^{i\gamma} e^{-i\xi x} d\xi \\ &= \frac{1}{2\pi} \sum_{k=-\infty}^\infty a_k c_\gamma \left[\int_{-k-1/2}^{-k+1/2} c_\gamma \operatorname{sgn}(\xi) |k|^{i\gamma} e^{-i\xi x} d\xi \right. \\ &\quad \left. + \int_{-k-1/2}^{-k+1/2} \operatorname{sgn}(\xi) (|\xi|^{i\gamma} - |k|^{i\gamma}) e^{-i\xi x} d\xi \right] \\ &= \frac{1}{2\pi} \sum_{k=-\infty}^\infty a_k c_\gamma \operatorname{sgn}(k) |k|^{i\gamma} e^{ikx} \left(\frac{\sin(x/2)}{x/2} \right) + \varepsilon(x, \gamma), \end{aligned}$$

where $\varepsilon(x, \gamma) = \frac{1}{2\pi} \sum_{k=-\infty}^\infty a_k c_\gamma \int_{-k-1/2}^{-k+1/2} \operatorname{sgn}(\xi) (|\xi|^{i\gamma} - |k|^{i\gamma}) e^{-i\xi x} d\xi$. Thus

$$H_\gamma f(x) = \left(\frac{\sin(x/2)}{x/2} \right) \tilde{T}_\gamma \varphi(x) + \varepsilon(x, \gamma).$$

However,

$$|\varepsilon(x, \gamma)| \leq \frac{|c_\gamma|}{2\pi} \sum_{k=-\infty}^{\infty} |a_k| \int_{-k-1/2}^{-k+1/2} ||\xi|^{i\gamma} - |k|^{i\gamma}| d\xi.$$

By the mean value theorem, we can estimate the integral in this expression by

$$\sup_{\xi \in (k-1/2, k+1/2)} |(e^{i\gamma \ln \xi})'| \frac{1}{2} \leq \frac{\gamma}{|k| - 1/2} \left(\frac{1}{2}\right).$$

Consequently,

$$\begin{aligned} |\varepsilon(x, \gamma)| &\leq \frac{|c_\gamma|}{4\pi} \sum_{k=-\infty}^{\infty} |a_k| \frac{\gamma}{|k| - 1/2} \\ &\leq \gamma \frac{|c_\gamma|}{4\pi} \left(\sum_{k=-\infty}^{\infty} |a_k|^2 \right)^{1/2} \left(\sum_{k=-\infty}^{\infty} \frac{1}{(|k| - 1/2)^2} \right)^{1/2} = O(\gamma). \end{aligned}$$

We now know

$$H_\gamma f(x) = \left(\frac{\sin(x/2)}{x/2} \right) \tilde{T}_\gamma \phi(x) + \varepsilon(x, \gamma), \quad \text{with } \varepsilon(x, \gamma) = O(\gamma).$$

For each $x \in E$, we know $\lim_{n \rightarrow \infty} \tilde{T}_{\gamma_n} \phi(x)$ will fail to exist. Since $\lim_{n \rightarrow \infty} \varepsilon(x, \gamma) = 0$, we know that $\lim_{n \rightarrow \infty} H_{\gamma_n} f(x)$ must also fail to exist on E .

It remains to show that $f \in L^p(\mathbb{R})$, $1 < p \leq \infty$. To see this we compute f explicitly, and find that

$$f(x) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} a_k e^{ikx} \left(\frac{\sin x/2}{x/2} \right).$$

Consequently

$$\|f\|_p^p = \left(\frac{1}{2\pi} \right)^p \sum_{n=-\infty}^{\infty} \int_{2\pi n}^{2\pi(n+1)} \left| \frac{\sin x/2}{x/2} \right|^p \left| \sum_{k=-\infty}^{\infty} a_k e^{ikx} \right|^p dx.$$

First note that $\left| \frac{\sin x/2}{x/2} \right|$ is bounded by $2/n$ on $[2\pi n, 2\pi(n+1)]$ if $n \neq 0$, or -1 , and by 1 on $[-2\pi, 0]$ and $[0, 2\pi]$. Also note that

$$\int_{2\pi n}^{2\pi(n+1)} \left| \sum_{k=-\infty}^{\infty} a_k e^{ikx} \right|^p dx$$

is a constant b_p independent of n . Consequently

$$\|f\|_p^p \leq \left(\frac{1}{2\pi} \right)^p \left(2 + \sum_{\substack{n=-\infty \\ n \neq -1, 0}}^{\infty} \frac{2^p}{|n|^p} \right) b_p < \infty. \quad \square$$

Remark. Theorem 2 can be strengthened to give a function that is in all $L^p(\mathbb{R})$, $0 < p \leq \infty$, such that almost everywhere convergence of $\{H_{\gamma_n} f\}$ fails. The idea is to replace translates of the characteristic function of $(-1/2, 1/2)$ by a suitably smooth “bump function.” See [1] for details.

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