REMARKS ON QUASICONVEXITY AND STABILITY OF EQUILIBRIA FOR VARIATIONAL INTEGRALS

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ABSTRACT. Let $F: \mathbf{R}^N \rightarrow \mathbf{R}$ be a uniformly strictly quasiconvex function (see [3, 4]) of class $C^{2+\alpha}$, $(0 < \alpha < 1)$, and be of polynomial growth. Then every smooth solution of the Euler-Lagrangian equation of the multiple integral $I(u; \Omega) = \int_{\Omega} F(Du(x)) dx$ is a minimum of I for variations of sufficiently small supports contained in Ω .

This note establishes the stability of solutions to the equilibrium equations for variational integrals under the constitutive assumption of uniformly strictly quasiconvexity. We show that, for a class of quasiconvex integrands, all equilibria are strong local minimizers of sufficiently small support. This work could be compared with that of Sivaloganathan [9] and Zhang [10], where the similar problems are studied under the constitutive assumption of polyconvexity.

Let $\Omega \subset \mathbf{R}^n$ be bounded and open. To any given map $u: \Omega \mapsto \mathbf{R}^N$, we associate an energy

$$(1) \quad I(u; \Omega) = \int_{\Omega} F(Du(x)) dx.$$

It is well known that any smooth minimizer of I satisfies the corresponding Euler-Lagrange equations:

$$(2) \quad \frac{\partial}{\partial x^\alpha} \left[\frac{\partial F}{\partial P_\alpha^i}(Du(x)) \right] = 0, \quad \text{for all } x \in \Omega, i = 1, 2, \dots, N.$$

We define the set of admissible maps

$$(3) \quad A_R(x_0) = \{u \in W^{1,p}(B_R(x_0); \mathbf{R}^N): u|_{\partial\Omega} = u_0|_{\partial\Omega}, B_R(x_0) \subset \subset \Omega\}$$

and consider the question of whether a given solution u_0 of (2) is a strong local minimizer of I , in the sense that u_0 minimizes I in $A_R(x_0)$ for some $x_0 \in \Omega$ and for some $R > 0$ (where we use $B_R(x_0)$, to denote the ball in $\mathbf{R}^n$ centered at x_0 with radius $R > 0$). We study this question in the case where the integrand F is uniformly strictly quasiconvex and of class $C^{2,\alpha}$.

This problem has been studied by many authors (see, e.g., Cesari [2], Rund [8] in the case that F is strictly convex and may depend on x , u as well, Sivaloganathan [9] for polyconvexity case, and the references therein). However, in

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the present case nonuniqueness phenomenon of solutions may occur (see Ball [1], John [6], Knops and Stuart [7]). It is interesting to study the behaviour of other solutions of (2) than the minimizers of (1).

Throughout this paper, summation convention is applied. For a function $F: \mathbf{R}^N \mapsto \mathbf{R}$, denote by $DF(P)$, $D^2F(P)$ the first and the second order derivatives of F while $\|\cdot\|_C$ denotes the supremum norm on the space of continuous functions on some $\bar{B}_R(x) \subset \Omega$. We use various C to denote positive constants independent of the variables.

Theorem. Suppose $F: \mathbf{R}^N \mapsto \mathbf{R}$ is of class $C_{\text{loc}}^{2,\alpha}$ with $0 < \alpha \leq 1$ and satisfies

(1) for some $p \geq 2 + \alpha$,

$$|D^2F(P + Q) - D^2F(P)| \leq C_1(1 + |P|^{p-2-\alpha} + |Q|^{p-2-\alpha})|Q|^\alpha;$$

(2) (uniformly strictly quasiconvexity (see Evans [3], Giaquinta and Modica [5], Fusco and Hutchinson [4])). For every open bounded set $G \subset \mathbf{R}^n$, every $P \in \mathbf{R}^N$ and every $\phi \in W_0^{1,p}(G; \mathbf{R}^N)$,

$$\int_G [F(P) + \nu(|D\phi|^p + |D\phi|^2)] dx \leq \int_G F(P + D\phi) dx,$$

where $\nu > 0$ is a constant.

Then, for every C^2 solution u_0 of (2) and each $x_0 \in \Omega$, there exists an $R > 0$ with $B_R(x_0) \Subset \Omega$, such that u_0 is a strong minimizer of $I(\cdot; B_R(x_0))$ on $A_R(x_0)$.

Remark. It is easy to see that (1) implies

$$|F(P)| \leq C(1 + |P|^p)$$

for some $C > 0$.

In general, even the minimizers of (1) are only partially regular (see e.g., [3, 4, 5]), i.e., there exists an open subset Ω_0 of Ω with $\text{meas}(\Omega \setminus \Omega_0) = 0$, such that $u \in C^{1,\alpha}(\Omega_0; \mathbf{R}^N)$, $0 < \alpha < 1$. Therefore, for partially regular solutions of (2), we conclude that u_0 is locally stable on Ω_0 , i.e., u_0 is a minimizer on $A_R(x_0)$ for some $R > 0$, $B_R(x_0) \subset \subset \Omega_0$.

Proof of the theorem. We prove the theorem by contradiction. For a fixed $x_0 \in \Omega$, if the conclusion of the theorem is not true, then there exists a sequence of positive R_j with $R_j \rightarrow 0$ as $j \rightarrow \infty$, such that $B_{R_j}(x_0) \Subset \Omega$ and u_0 is not a minimizer in $A_{R_j}(x_0)$. Hence there exists a sequence of functions $\phi_j \in W_0^{1,p}(B_{R_j}(x_0))$ such that

$$(4) \quad \int_{B_{R_j}(x_0)} F(Du_0 + D\phi_j) dx < \int_{B_{R_j}(x_0)} F(Du_0) dx.$$

Since u_0 is a solution of (2), we have, from (4),

$$(5) \quad \begin{aligned} 0 &> \int_{B_{R_j}(x_0)} [F(Du_0 + D\phi_j) - F(Du_0)] dx = \int_{B_{R_j}(x_0)} DF(Du_0) D\phi_j dx \\ &+ \int_{B_{R_j}(x_0)} \int_0^1 (1-t) D^2F(Du_0 + tD\phi_j) D\phi_j D\phi_j dt dx \end{aligned}$$

and notice that, from the divergence theorem,

$$\int_{B_{R_j}(x_0)} DF(Du_0)D\phi_j \, dx = 0,$$

so that, in (5), we have

$$(6) \quad 0 < - \int_{B_{R_j}(x_0)} \int_0^1 (1-t) D^2 F(Du_0 + tD\phi_j) D\phi_j D\phi_j \, dt \, dx.$$

On the other hand, from (2),

$$\begin{aligned} & \nu \int_{B_{R_j}(x_0)} (|D\phi_j|^2 + |D\phi_j|^p) \, dx \\ & \leq \int_{B_{R_j}(x_0)} [F(Du_0(x_0)) + D\phi_j] - F(Du_0(x_0)) \, dx \\ (7) \quad & = \int_{B_{R_j}(x_0)} DF(Du_0(x_0)) D\phi_j \, dx \\ & \quad + \int_{B_{R_j}(x_0)} \int_0^1 (1-t) D^2 F(Du_0(x_0)) + tD\phi_j) D\phi_j D\phi_j \, dt \, dx. \end{aligned}$$

Still from the divergence theorem, we have

$$\int_{B_{R_j}(x_0)} DF(Du_0(x_0)) D\phi_j \, dx = 0,$$

so that adding (6) to (7) gives

$$\begin{aligned} & \nu \int_{B_{R_j}(x_0)} (|D\phi_j|^p + |D\phi_j|^2) \, dx \\ & < \int_{B_{R_j}(x_0)} \int_0^1 (1-t) [D^2 F(Du_0(x_0) + tD\phi_j) \\ & \quad - D^2 F(Du_0(x) + tD\phi_j)] D\phi_j D\phi_j \, dt \, dx \\ (8) \quad & \leq \int_{B_{R_j}(x_0)} C[1 + |Du_0(x)|^{p-2-\alpha} + |Du_0(x_0)|^{p-2-\alpha} \\ & \quad + |D\phi_j|^{p-2-\alpha}] |Du_0(x) - Du_0(x_0)|^\alpha |D\phi_j|^2 \, dx \\ & \leq C(\|Du_0\|_C) \int_{B_{R_j}(x_0)} |Du_0(x_0) - Du_0(x)|^\alpha (1 + |D\phi_j|^{p-2}) |D\phi_j|^2 \, dx \\ & \leq C \max_{x \in B_{R_j}} \{|Du_0(x_0) - Du_0(x)|^\alpha\} \int_{B_{R_j}(x_0)} (|D\phi_j|^p + |D\phi_j|^2) \, dx. \end{aligned}$$

Dividing both sides of (8) by $\int_{B_{R_j}(x_0)} (|D\phi_j|^p + |D\phi_j|^2) \, dx$ we obtain

$$(9) \quad \nu < C \max_{x \in B_{R_j}(x_0)} \{|Du_0(x_0) - Du_0(x)|^\alpha\}.$$

Passing to the limit $j \rightarrow \infty$, we have $R_j \rightarrow 0$ and

$$\max_{x \in B_{R_j}(x_0)} \{|Du_0(x_0) - Du_0(x)|\} \rightarrow 0$$

so that

$$\nu \leq \lim_{j \rightarrow \infty} C \max_{x \in B_{R_j}(x_0)} \{|Du_0(x_0) - Du_0(x)|^\alpha\} = 0.$$

This contradicts to the assumption $\nu > 0$.

Remark. This result could be easily extended to the case where F depends explicitly on x , say, $F = F(x, P)$. It seems to be unknown whether general weak equilibria of (1) in $W^{1,p}(\Omega; \mathbf{R}^N)$ are local minima for variations of sufficiently small support near a Lebesgue point of Du .

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