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# REGULAR STATES AND COUNTABLE ADDITIVITY ON QUANTUM LOGICS

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ABSTRACT. We give a counterexample of the result of Béaver and Cook concerning a generalization of the Alexandroff theorem for regular, finitely-additive states on quantum logics using states on the system of all splitting subspaces of an incomplete inner-product space. Moreover, we introduce another type of state regularity which entails countable additivity of states on logics.

## 1. INTRODUCTION

We recall that a *quantum logic* is a poset $L$ with the minimal and maximal elements 0 and 1, respectively, and with the unary operation (named orthocomplementation) $\perp : L \to L$ such that

(i) $(a^\perp)^\perp = a$, for any $a \in L$;
(ii) if $a \leq b$, then $b^\perp \leq a^\perp$;
(iii) $a \vee a^\perp = 1$, for any $a \in L$;
(iv) if $a \leq b^\perp$, then $a \vee b \in L$; and
(v) if $a \leq b$, then $b = a \vee (b \wedge a^\perp)$ (orthomodular property).

($\vee$ and $\wedge$ denote the operations sup and inf.) We note that, in view of (i)–(iv), if $a \leq b$, then $b \wedge a^\perp$ exists in $L$. We say that two elements $a$ and $b$ of $L$ are *orthogonal*, denoted $a \perp b$, if $a \leq b^\perp$. If $L$ has the property that any sequence $\{a_n\}$ of mutually orthogonal elements of $L$ has a supremum, $\bigvee_{n=1}^{\infty} a_n$, in $L$, then $L$ is called a *$\sigma$-quantum logic.*

A *Boolean algebra* is a poset $\mathscr{B}$ containing the minimal and maximal elements 0 and 1, respectively, such that $\mathscr{B}$ is equipped with the operation of complementation $\perp$ satisfying (i) and (ii) of the definition of quantum logic and has the additional properties

(L) $\qquad a \vee b \in \mathscr{B}$ for any $a, b \in \mathscr{B}$ (lattice property)

(so that any nonempty finite subset of $\mathscr{B}$ has supremum and infimum)

(D) $\qquad (a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ (distributive property).

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It is well known that every Boolean algebra is isomorphic to some algebra of sets.

A nonempty subset $\mathscr{B}$ of a quantum logic $L$ is a *Boolean subalgebra* of $L$ if (i) $0, 1 \in \mathscr{B}$; (ii) $a \in \mathscr{B}$ implies $a^\perp \in \mathscr{B}$; and (L) and (D) hold in $\mathscr{B}$.

A *state* (more precisely, a finitely-additive state) on a quantum logic $L$ is a mapping $m: L \to [0, 1]$ such that (i) $m(1) = 1$; (ii) $m(a \vee b) = m(a) + m(b)$ whenever $a \perp b$. A state $m$ is *countably additive* if $m(\bigvee_{n=1}^\infty a_n) = \sum_{n=1}^\infty m(a_n)$ whenever $\{a_n\}_{n=1}^\infty$ is a sequence of mutually orthogonal elements and $\bigvee_{n=1}^\infty a_n \in L$. A state $m$ is *completely additive* if $m(\bigvee_{t\in T} a_t) = \sum_{t\in T} m(a_t)$ whenever $\{a_t : t \in T\}$ is a system of mutually orthogonal elements of $L$ and $\bigvee_{t\in T} a_t \in L$.

In the present paper, we show that the proof of Béaver and Cook [2] on regular states contains a gap, and we present an example of a regular, finitely-additive state that is not countably additive. On the other hand, we give a new type of state regularity that will entail the countable additivity.

## 2. Regular states

Let $\mathscr{P}$ be a nonvoid subset of a quantum logic $L$. A state $m$ is called *$\mathscr{P}$-regular* (more precisely, *$\mathscr{P}$-regular in the sense of Béaver and Cook*), if for each $\varepsilon > 0$ and each $b \in L$ there exists an $a \in \mathscr{P}$ with $a \leq b$ and $m(b \wedge a^\perp) < \varepsilon$.

One of the most important examples of a quantum logic is the system $L(H)$ of all closed subspaces of a real or complex Hilbert space $H$, which is a complete lattice and plays a considerable role in the axiomatic model of quantum mechanics (see, e.g., [15]). The famous theorem of Gleason [8] asserts that any countably-additive state $m$ on $L(H)$, $3 \leq \dim H \leq \aleph_0$, is of the following form

$$m(M) = \operatorname{tr}(TP^M), \qquad M \in L(H), \tag{2.1}$$

where $T$ is a positive operator of the trace class on $H$ and $P^M$ is the orthogonal projector from $H$ onto $M$.

More generally, let $S$ be a real or complex inner-product space with an inner product $(\cdot, \cdot)$. By a subspace of $S$ we shall understand a linear closed subspace of $S$. For any subspace $M$ of $S$, $M^\perp$ denotes the set of all $x \in S$ such that $(x, y) = 0$ for all $y \in M$. We denote by $E(S)$ the set of all subspaces $M$ of $S$ such that $M + M^\perp = S$. Then $L = E(S)$ is a quantum logic which contains any complete and, therefore, any finite-dimensional subspace. Moreover, $E(S)$ is a $\sigma$-quantum logic iff $S$ is complete [6].

Let $\mathscr{P}^\perp = \{a^\perp : a \in \mathscr{P}\}$. An element $b \in \mathscr{P}$ is called *finitely coverable* if, for any sequence $\{a_1^\perp, a_2^\perp, \dots\} \subseteq \mathscr{P}^\perp$ such that $\bigvee_{k=1}^\infty a_k^\perp$ exists in $L$ and $b \leq \bigvee_{k=1}^\infty a_k^\perp$, there is an integer $n$ such that $\bigvee_{k=1}^n a_k^\perp$ exists in $L$ and $b \leq \bigvee_{k=1}^n a_k^\perp$. $\mathscr{P}$ is called *finitely coverable* if each element of $\mathscr{P}$ is finitely coverable.

Béaver and Cook [2] presented the following result: Let $L$ be a $\sigma$-quantum logic and $\mathscr{P} \subseteq L$ be finitely coverable such that $\mathscr{P}^\perp$ contains the join of any sequence in $\mathscr{P}^\perp$. Then any $\mathscr{P}$-regular state on $L$ is countably additive.

Unfortunately their proof is incorrect, because they used the subadditivity of a state (i.e., $m(a) \leq \sum_{i=1}^n m(a_i)$ if $a \leq \bigvee_{i=1}^n a_i$), which is invalid, in general, in quantum logics (consider, for example, a state of the form (2.1)). Also, the assumption that $L$ is a $\sigma$-quantum logic was not used in the proof. If $m$ is subadditive—for example, if $L$ is a Boolean algebra—their proof works.

Below we present an example of a quantum logic $L$, a finitely-coverable subset $\mathscr{P} \subseteq L$ such that $\mathscr{P}^\perp$ contains the join of any sequence in $\mathscr{P}^\perp$, and a $\mathscr{P}$-regular state $m$ on $L$ that is not countably additive. In particular, it shows that $\mathscr{P}$-regularity is not a sufficient condition for countable additivity.

**Counterexample 2.1.** Let $S$ be an inner-product space. For any $x \in S$, $\|x\| = 1$, the mapping $m_x$ on $E(S)$ defined via

$$m_x(M) = \|x_M\|^2, \qquad M \in E(S), \tag{2.2}$$

if $x = x_M + x_{M^\perp}$, where $x_M \in M$, $x_{M^\perp} \in M^\perp$, is a state on the quantum logic $L = E(S)$. The system $\mathscr{P} = \mathscr{P}(S)$ of all finite-dimensional subspaces is finitely coverable. If $S$ is a separable, incomplete inner-product space, then any $m_x$ is a $\mathscr{P}$-regular state which is not countably additive.

*Proof.* If $M$ and $N$ are mutually orthogonal, splitting subspaces of $S$, then $M + N = M \vee N \in E(S)$ (see, e.g., [9, 5]). Hence, it is simple to verify that any $m_x$ is a state of a quantum logic $L = E(S)$.

Now we show that $\mathscr{P} = \mathscr{P}(S)$ is finitely coverable. In fact, any $M \in E(S)$ is finitely coverable by $\mathscr{P}^\perp$. To see this, let $M \in E(S)$ and let $\{M_1^\perp, M_2^\perp, \dots\}$ be a sequence in $\mathscr{P}^\perp$ such that $M \subseteq \bigvee_{k=1}^\infty M_k^\perp$ (the latter join belongs always to $E(S)$). Each $M_k^\perp$ is a subspace of finite codimension. If the integer $n_k$, $k = 1, 2, \dots$, is the codimension of $M_1^\perp \vee \cdots \vee M_k^\perp \in E(S)$, then $n_1 \geq n_2 \geq \cdots \geq 0$. Thus, for some $i$, $n_i = n_{i+1} = \cdots$ and $M \leq \bigvee_{k=1}^i M_k^\perp$.

Suppose that $S$ is separable and incomplete. It is straightforward to show that $m_x$ is of the form

$$m_x(M) = \|P^{\overline{M}} x\|^2, \qquad M \in E(S), \tag{2.3}$$

where $P^{\overline{M}}$ is the orthoprojector from the completion $\overline{S}$ of $S$ onto the completion $\overline{M}$ of $M$. The separability of $S$ entails the existence of an orthonormal basis (ONB) $\{x_n\}$ in any splitting subspace $M$ of $S$. It is simple to show that $\{x_n\}$ is an ONB in $\overline{M}$, too. Therefore, $m_x(M) = \|P^{\overline{M}} x\|^2 = \|\sum_n P_{x_n} x\|^2 = \sum_n \|P_{x_n} x\|^2$, where $P_u$ is an orthogonal projection onto one-dimensional subspace spanned by a nonzero vector $u \in S$.

Given $\varepsilon > 0$, we can find a finite-dimensional subspace $N = \mathrm{sp}(x_1, \dots, x_n) \subseteq M$, where sp denotes the span over $x_1, \dots, x_n$ such that $m_x(M \cap N^\perp) < \varepsilon$.

Since $S$ is incomplete and separable, there is a maximal orthonormal set $\{x_i\}_{i=1}^\infty$ in $S$ that is not a basis (see, for example [11]). Consequently, there is a $z \in S$ such that $1 = \|z\|^2 \neq \sum_{i=1}^\infty |(z, x_i)|^2$. Therefore, $m_z(S) = \|z\|^2 \neq \sum_{i=1}^\infty |(z, x_i)|^2 = \sum_{i=1}^\infty m_z(P_{x_i})$, although $\bigvee_{i=1}^\infty P_{x_i} = S$, and $m_z$ is not countably additive. Actually, in this case $E(S)$ does not possess any countably-additive states, as a consequence of the result of [6] saying that $S$ is complete iff $E(S)$ has at least one countably-additive state (completely additive for general $S$). Therefore, any of the states $m_x$ is a $\mathscr{P}$-regular state but not countably additive. Q.E.D.

On the other hand, we show below that on a very important quantum logic, $L(H)$ of a Hilbert space $H$, the assertion of Béaver and Cook is correct, even when a finitely-additive state is not subadditive.

**Theorem 2.2.** *A finitely-additive state $m$ on $L(H)$, $\dim H \geq 3$, is of the form* (2.1) *iff $m$ is $\mathscr{P}(S)$-regular* (*in the sense of Béaver and Cook*).

*Proof*. Suppose that $m$ is of the form (2.1). Then $m$ is completely additive. Let $M$ be an arbitrary element of $L(H)$ and an ONB $\{f_t\}$ in $M$. Due to the complete additivity of $m$, there is a sequence $\{f_n\} \subseteq \{f_t\}$ such that $m(M) = \sum_n m(P_{f_n})$. For $\varepsilon > 0$, there exists an integer $k$ sufficiently large for which $N = \mathrm{sp}(f_1, \ldots, f_k)$ has the property $m(M \cap N^\perp) = m(M) - m(N) = \sum_{n=k+1}^{\infty} m(P_{f_n}) < \varepsilon$.

Now let $m$ be a $\mathscr{P}(H)$-regular (in the sense of Béaver and Cook), finitely-additive state on $L(H)$. Due to the result by Aarnes [1, Proposition 2, p. 609], any finitely-additive state on $L(H)$ is uniquely decomposed into the sum $m = m_1 + m_2$, where $m_1$ is a completely-additive state and $m_2$ is a finitely-additive state that vanishes on finite-dimensional subspaces of $H$. Due to the Maeda theorem [12] or Aarnes [1], $m_1$ is of the form (2.1) for some positive operator of trace class on $H$. The $\mathscr{P}(H)$-regularity of $m$ and $m_1$ entails $m_2 = 0$. Thus, if $m$ is a $\mathscr{P}(H)$-regular state on $L(H)$, then $m = m_1$ and $m$ is of the form (2.1). Q.E.D.

A system of states, $\mathscr{M}$, of a quantum logic $L$ is full if the following condition is satisfied: If $m(a) \leq m(b)$ for all $m \in \mathscr{M}$, then $a \leq b$.

**Example 2.3.** There is a lattice $\sigma$-quantum logic $L$ with a full system of countably-additive states on $L$, a subquantum logic $L_0$ of $L$ that is not a lattice, and a finitely-coverable system $\mathscr{P} \subseteq L_0$ such that the restriction of any $m \in \mathscr{M}$ onto $L_0$ is a $\mathscr{P}$-regular state which is not countably additive.

*Proof*. Let $S$ be an incomplete, separable inner-product space. Put $L = L(\overline{S})$, and for any $M \in E(S)$ define $\varphi(M) = \overline{M} \in L$. Then

(i) if $M \neq N$, then $\varphi(M) \neq \varphi(N)$;
(ii) $\varphi(M \vee N) = \varphi(M) \vee \varphi(N)$ if $M \perp N$; and
(iii) $\varphi(M^\perp) = \varphi(M)^{\perp \overline{S}} = \{x \in \overline{S} : (x, y) = 0 \text{ for each } y \in \varphi(M)\}$.

If we put $L_0 = \{\varphi(M) : M \in E(S)\}$, then $L_0$ is a subquantum logic of $L$ that is isomorphic to $E(S)$. The system $\mathscr{M} = \{w_x : x \in S, \|x\| = 1\}$, where $w_x$ is a mapping on $L$ defined in a manner analogous to $m_x$ in (2.2), is a full system of countably-additive states on $L$. Indeed, let $w_x(M) \leq w_x(N)$, $x \in S$, $\|x\| = 1$, then we have $w_x(M) = (P^M x, x)$, $M \in L(\overline{S})$, where $P^M$ is the orthoprojector from $\overline{S}$ onto $\overline{M}$. Therefore, for all vectors $x$ from $S$ we have $(P^M x, x) \leq (P^N x, x)$, so that $(P^M x, x) \leq (P^N x, x)$ for all $x \in \overline{S}$; i.e., $M \subseteq N$.

A finitely-coverable system $\mathscr{P}$ is defined as $\mathscr{P} = \{\varphi(M) : M \in E(S), \dim M < \infty\}$. Following the lines of Counterexample 2.1 and noting that $w_x(\varphi(M)) = m_x(M)$ for each $M \in E(S)$, we see that $w_x | L_0$ is $\mathscr{P}$-regular but is not countable additive. Q.E.D.

## 3. Alexandroff's theorem on quantum logics

Now we introduce another type of state regularity that will imply countable additivity. Let $\mathscr{P}$ be a nonvoid subset of a quantum logic $L$. We say that a state $m$ is *$\mathscr{P}$-regular* if, for every sequence $\{q_n\}_{n=1}^{\infty}$ of mutually orthogonal elements of $L$ such that $q = \bigvee_{n=1}^{\infty} q_n$ exists in $L$, there is a block $\mathscr{B} \subseteq L$ (i.e., a maximal Boolean subalgebra of $L$) such that for each $\varepsilon > 0$ and every $r \in \{q, q_1^{\perp}, q_2^{\perp}, \dots\}$, there exists a $p \in \mathscr{B} \cap \mathscr{P}$ with $p \leq r$ and $m(r \wedge p^{\perp}) < \varepsilon$.

It is evident that any $\mathscr{P}$-regular state is a $\mathscr{P}$-regular state in the sense of Béaver and Cook. The converse assertion is not true, in general, as we shall see below.

If $L$ is a Boolean algebra, then both notions coincide.

**Theorem 3.1.** *Let $L$ be a quantum logic, $\mathscr{P} \subseteq L$ be finitely coverable, and $\mathscr{P}^{\perp}$ contain the join of any sequence in $\mathscr{P}^{\perp}$. Then any $\mathscr{P}$-regular state $m$ on $L$ is countably additive.*

*Proof.* The proof is similar to that in [2]. Let $\{q_n\}_{n=1}^{\infty}$ be an orthogonal sequence in $L$ with $q = \bigvee_{n=1}^{\infty} q_n$ in $L$. Then $q \geq \bigvee_{i=1}^{n} q_i$ for all $n \geq 1$ and $m(q) \geq \sum_{i=1}^{n} m(q_i)$, so that $m(q) \geq \sum_{i=1}^{\infty} m(q_i)$. For $\{q_n\}_{n=1}^{\infty}$ there is a block $\mathscr{B}$ of $L$ such that, for any $\varepsilon > 0$, there is a sequence $\{p_1, p_2, \dots\} \subseteq \mathscr{B} \cap \mathscr{P}$ with $p_n \leq q_n^{\perp}$ and $m(q_n^{\perp} \wedge p_n^{\perp}) < \varepsilon/2^n$. Also, there exists a $p \in \mathscr{B} \cap \mathscr{P}$ with $p \leq q$ and $m(q \wedge p^{\perp}) < \varepsilon$. Thus, $p^{\perp} \geq q^{\perp} = \bigwedge_{i=1}^{\infty} q_i^{\perp} \geq \bigwedge_{i=1}^{\infty} p_i$, so $p \leq \bigvee_{i=1}^{\infty} p_i^{\perp}$. Since $\mathscr{P}$ is finitely coverable, there exists an integer $k$ such that $p \leq \bigvee_{i=1}^{k} p_i^{\perp}$. Also, we have $m(p_n^{\perp}) = m(q_n) + m(q_n^{\perp} \wedge p_n^{\perp})$, and this implies $m(p_n^{\perp}) - \varepsilon/2^n \leq m(q_n)$. Similarly, $m(p) + \varepsilon \geq m(q)$. Using the subadditivity of $m$ in Boolean subalgebras, we conclude that

$$\begin{aligned}\sum_{n=1}^{\infty} m(q_n) &\geq \sum_{n=1}^{\infty} m(p_n^{\perp}) - \varepsilon \geq \sum_{n=1}^{k} m(p_n^{\perp}) - \varepsilon \geq m(p) - \varepsilon \\ &\geq m(q) - 2\varepsilon.\end{aligned}$$

Therefore, $m(q) = \sum_{n=1}^{\infty} m(q_n)$. Q.E.D.

Let $L = L(H)$ be the quantum logic of a Hilbert space $H$, and $\mathscr{P} = \mathscr{P}(H)$ be the system of all finite-dimensional subspaces of $H$. Then any countably-additive state $m$ on $L$ of the form $m(M) = \operatorname{tr}(TP^M)$, $M \in L(H)$, where $T$ is a positive operator of trace equal to $1$, is $\mathscr{P}$-regular, and $m$, $\mathscr{P}$, $L$ satisfy the conditions of Theorem 3.1. Moreover, in view of Theorem 2.2, the $\mathscr{P}(H)$-regularity and the $\mathscr{P}(H)$-regularity (in the sense of Béaver and Cook) coincide on $L(H)$ if $\dim H \geq 3$. On the other hand, we recall that any countably additive state on $L(H)$ is of the form (2.1) iff the dimension of $H$ is a nonmeasurable cardinal $\neq 2$ ([4, 7]).

**Example 3.2.** Let $S$ be an incomplete, separable inner-product space. For any unit vector $x \in \overline{S}$ we define a mapping $m_x \colon E(S) \to [0, 1]$ via $m_x(M) = \|P^{\overline{M}} x\|^2$, $M \in E(S)$, and let $\mathscr{P}$ be the system of all finite-dimensional subspaces of $S$. Then $m_x$ is $\mathscr{P}$-regular in the sense of Béaver and Cook and not $\mathscr{P}$-regular. This follows from the result in [6] saying that $S$ is complete iff $E(S)$ possesses at least one countably additive state.

**Corollary 3.3.** *Let $S$ be of a countable orthogonal dimension (i.e., the cardinality of any maximal orthonormal system in $S$ is countable). $S$ is complete iff $E(S)$ possesses at least one $\mathscr{P}$-regular state, where $\mathscr{P}$ is the system of all finite-dimensional subspaces of $S$.*

*Proof.* This follows from Theorem 3.1 and the result in [6]. Q.E.D.

We note that according to [3, pp. 21, 38], the range of the observable corresponding to the momentum operator is a block in $L = L(H)$ that does not contain nonzero finite-dimensional subspaces. Therefore, not every block in $L(H)$ may be used for an approximation of a $\mathscr{P}(H)$-regular state.

## 4. Regularity on $\sigma$-classes

Now we exhibit the problem of the countable additivity of regular states on a special type of $\sigma$-quantum logics that are called $\sigma$-classes, and we present two results in this direction.

Let $X$ be a nonempty set. A *$\sigma$-class* $L$ of subsets of $X$ is a collection of subsets of $X$ that satisfy the following:

(i) $X \in L$;

(ii) if $E \in L$, then $E^c : X - E \in L$;

(iii) if $A_i \in L$, $i \geq 1$, are mutually disjoint, then $\bigcup_{i=1}^{\infty} A_i \in L$.

The set $L$ may be regarded as a partially ordered set, where the partial ordering is defined by the set-theoretical inclusion, and $A^{\perp} = A^c$. It is easy to check that $L$ is a $\sigma$-quantum logic, where sup and inf, $F \vee G$ and $F \wedge G$, respectively, are defined in the usual way relative to $L$. Note, however, that $F \vee G$ $(F \wedge G)$ need not equal $F \cup G$ $(F \cap G)$ even if the former exist in $L$; they are equal if the latter are in $L$.

In the following two examples, we show that the subadditivity does not hold, in general, relative neither to $\cup$ nor to $\vee$.

**Example 4.1** [10, p. 71]. Let $X = [0, 6]$ and $L = \{\varnothing, X, A, B, C, A^c, B^c, C^c\}$, where $A = [0, 4]$, $B = [2, 5]$, $C = [0, 1] \cup [2, 3] \cup [5, 6]$. Now we define the state $m$ on $L$ as follows:

$$m(\varnothing) = 0, \qquad m(A) = m(B) = m(C) = 1/4,$$
$$m(A^c) = m(B^c) = m(C^c) = 3/4, \qquad m(X) = 1.$$

Then $1 = m(X) = m(A \cup B \cup C) > 3/4 = m(A) + m(B) + m(C)$, so that $m$ is not subadditive.

**Example 4.2.** Let $X = \{1, 2, 3, 4\}$, $L = \{\varnothing, X, \{1, 2\}, \{3, 4\}, \{1, 3\}, \{2, 4\}\}$, $m(\varnothing) = 0$, $m(X) = 1$, $m(\{1, 2\}) = m(\{3, 4\}) = 1/2$, $m(\{1, 3\}) = 1/3$, $m(\{2, 4\}) = 2/3$. Then

$$1 = m(X) = m(\{1, 2\} \vee \{1, 3\}) > m(\{1, 2\}) + m(\{1, 3\}) = 5/6.$$

On the other hand, any $m$ satisfies the condition of subadditivity (with respect to the union $\cup$) for two sets $A, B \in L$ if $A \cup B \in L$ [10, p. 71].

Now we show that the proof of the result of Béaver and Cook works in the case of $\sigma$-classes in the "almost" original formulation. Namely, the following is true:

**Theorem 4.3.** *Let $L$ be a $\sigma$-class of subsets of a set $X \neq \varnothing$. Let $\mathscr{P} \subseteq L$ be finitely coverable* (*with respect to* $\vee$) *and contain the union of any sequence in it. Then any $\mathscr{P}$-regular state $m$* (*in the sense of Béaver and Cook*) *on $L$ is countably additive.*

*Proof*. We show that $m$ is $\mathscr{P}$-regular. Since $\mathscr{P}$ is closed with respect to any union of elements from $\mathscr{P}$, we conclude by [13] that there is a Boolean subalgebra $\mathscr{B} \subseteq L$ containing $\mathscr{P}$. Without loss of generality, we may assume that $\mathscr{B}$ is a block of question which is necessary for the validity of Theorem 3.1. Q.E.D.

A different approach to that of Alexandroff for a criterion of $\sigma$-additivity of a set function defined on a $\sigma$-algebra of subsets of a set $X$ is that of E. Marczewski [14]. In this case, no topology on $X$ is supposed. We show that such an approach may be applied to $\sigma$-classes.

A collection $\mathscr{K}$ of subsets of a set $X \neq \varnothing$ is said to be compact [14] if for any sequence $\{K_n\}_{n=1}^{\infty}$ of elements of $\mathscr{K}$ we have $K_1 \cap K_2 \cap \cdots \cap K_n \neq \varnothing$ for all $n \geq 1$, imply $\bigcap_{n=1}^{\infty} K_n \neq \varnothing$. Let $L$ be a $\sigma$-class of subsets of a set $X$ and $m$ be a state on $L$. We say that $m$ is compact (with respect to $\mathscr{K}$), provided that for any $E \in L$ and any $\varepsilon > 0$ there exist a $K \in \mathscr{K}$ and an $F \in L$, such that $E \supseteq K \supseteq F$ and $m(E \cap F^c) < \varepsilon$.

Denote $\widetilde{L} = \{E \cap F^c : E \in L,\ F \in L,\ E \supseteq F$ and there is a $K \in \mathscr{K}$ such that $E \supseteq K \supseteq F\}$.

**Theorem 4.4.** *Let $L$ be a $\sigma$-class of subsets of a set $X$. Let $\mathscr{K} \subseteq 2^X$ be compact, $m$ a compact state on $L$, and $L$ contain every finite union of elements from $\widetilde{L}$. Then $m$ is countably additive.*

*Proof.* It suffices to show that $\lim_n m(E_n) = 0$ for each decreasing sequence $\{E_n\}$ in $L$ such that $\bigcap_n E_n = \varnothing$. Let $\varepsilon > 0$, and for any $n \geq 1$, choose a $K_n \in \mathscr{K}$ such that $E_n \supseteq K_n \supseteq F_n$, where $F_n \in L$ and $m(E_n \cap F_n^c) < \varepsilon/2^n$. Evidently $\varnothing = \bigcap_{n=1}^{\infty} E_n \supseteq \bigcap_{n=1}^{\infty} K_n \supseteq \bigcap_{n=1}^{\infty} F_n$. Therefore, there is an integer $n_0$ such that, for $n > n_0$, we have $\bigcap_{i=1}^{n} K_i = \varnothing$; hence $\bigcap_{i=1}^{n} F_i = \varnothing$.

As in the proof of Theorem 4.3, we conclude that $\widetilde{L}$ is contained in a Boolean subalgebra of $L$, so that $m$ is subadditive on it. Therefore, for any $n > n_0$, we have

$$m(E_n) = m\left(\bigcap_{i=1}^{n} E_i\right) = m\left(\bigcap_{i=1}^{n} E_i - \bigcap_{i=1}^{n} F_i\right) \leq m\left(\bigcup_{i=1}^{n} (E_i \cap F_i^c)\right) < \varepsilon.$$

Thus, $\lim_n m(E_n) = 0$, which entails the $\sigma$-additivity of $M$ on $L$. Q.E.D.

*Remark.* Observe that in Theorem 4.4, $\mathscr{K}$ need not be contained in $L$.

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