ON THE MEAN CURVATURE ESTIMATES FOR BOUNDED SUBMANIFOLDS

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ABSTRACT. A Liouville-type theorem is proved for strongly subharmonic functions on complete riemannian manifolds of bounded curvature. We use this to give a simple proof of a theorem of Jorge, Koutroufiotis and Xavier, which gives an estimate for the exterior size of a submanifold in terms of the sup of the length of its mean curvature.

We give a short proof of the following theorem of Jorge and Xavier [3].

Theorem 1. *Let M and $\bar{M}$ be riemannian manifolds and let $f : M \rightarrow \bar{M}$ be an isometric immersion. Suppose that M is complete with $\inf$ scalar curvature $> -\infty$. Let $\delta := \sup$ of the sectional curvature of $\bar{M}$, $H := \sup$ of the length of mean curvature vector of f , and λ be such that there exists some closed normal ball B_λ^- of radius λ in $\bar{M}$ containing $f(M)$. Then,*

$$\lambda \geq \begin{cases} (1/\sqrt{-\delta}) \tanh^{-1}(\sqrt{-\delta}/H) & \text{if } \delta < 0, \\ 1/H & \text{if } \delta = 0, \\ \text{Min}\{(1/\sqrt{\delta}) \tan^{-1}(\sqrt{\delta}/H), \pi/(2\sqrt{\delta})\} & \text{if } \delta > 0. \end{cases}$$

Theorem 1 follows from

Theorem 2. *Let M be a complete riemannian manifold with bounded sectional curvature and θ a positive constant. Then, every C^2 solution to the inequality $\Delta u \geq \theta$ is unbounded.*

Proof. First, we claim that given any $r > 0$, there exists $\alpha > 0$ such that for any $p \in M$, we can construct a C^2 function $\tilde{v}_{p,r} : M \rightarrow \mathcal{R}_+$ such that $\tilde{v}_{p,r}(p) = 1$, $\tilde{v}_{p,r}$ vanishes outside $B_r(p)$, and $|\text{Hess}_x \tilde{v}_{p,r}| \leq \alpha$ for all $x \in M$. That such a $\tilde{v}_{p,r}$ exists independent of the injectivity radius of M follows by smoothing a bump function by convolution in the tangent space using the techniques of Theorem 1.8 in [1]. Set $v_{p,r} := \theta \tilde{v}_{p,r}/(\alpha\sqrt{d})$, where $d := \dim M$. Then, $|\text{Hess } v_{p,r}| \leq \theta/\sqrt{d}$.

Now, assume that u is bounded. Let $n := \sup u$. Then for $r > 0$ fixed, there exists a point $q \in M$ such that $n - u(q) < \theta/(\alpha\sqrt{d})$ where α is as in the

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construction above. Then,

$$\Delta(u + v_{q,r}) \geq \Delta u - |\Delta v_{q,r}| \geq \Delta u - \frac{|\text{Hess } v_{q,r}|}{\sqrt{d}} \geq \theta - \theta = 0,$$

so the function $u + v_{q,r}$ is subharmonic on M . On the other hand, $u(q) + v_{q,r}(q) > n - \theta/(\alpha\sqrt{d}) + \theta\tilde{v}_{q,r}(q)/(\alpha\sqrt{d}) = n$ while $u + v_{q,r}$ coincides with u outside $B_r(q)$. Therefore, $u + v_{q,r}$ must attain some maximum point in the set $B_r(q)$, contradicting the maximum principle.

Proof of Theorem 1. By the assumption on the scalar curvature and the Gauss equation, the sectional curvature of M is bounded (cf. [2, p. 722]). Assume that there exists a point $o \in \overline{M}$ such that $f(M) \subset B_\lambda(o)$. If $\delta > 0$, assume also that $\lambda < \pi/(2\sqrt{\delta})$. Then, it is well known (cf. [2, 3]) that the function $\varphi : M \rightarrow \mathcal{R}_+$ defined by $\varphi(x) := \text{dist}_{\overline{M}}(x, o)^2$ satisfies $\Delta\varphi(x) \geq d(\varepsilon - H\lambda)$ where

$$\varepsilon := \begin{cases} \lambda\sqrt{-\delta}\coth(\lambda\sqrt{-\delta}) & \text{if } \delta < 0, \\ 1 & \text{if } \delta = 0, \\ \lambda\sqrt{\delta}\cot(\lambda\sqrt{\delta}) & \text{if } \delta > 0. \end{cases}$$

Hence substituting, if λ is less than the prescribed constant, $\Delta\varphi \geq \theta := \sqrt{d}\varepsilon$. By Theorem 1, φ is unbounded, which is absurd.

REFERENCES

1. J. Cheeger and M. Gromov, *Chopping riemannian manifolds*, preprint.
2. L. Jorge and D. Koutroufiotis, *An estimate for the curvature of bounded submanifolds*, Amer. J. Math. **103** (1981), 711–725.
3. L. Jorge and F. Xavier, *An inequality between the exterior diameter and the mean curvature of bounded immersions*, Math. Z. **178** (1981), 77–82.

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