DIMENSIONS OF TOPOLOGICAL GROUPS CONTAINING THE BOUQUET OF TWO CIRCLES

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(Communicated by Dennis Burke)

ABSTRACT. In this paper we prove the following: If a topological group G contains the bouquet $S^1 \vee S^1$, then $\dim G \geq 2$ holds. This is a counterexample to a question of Bel'nov in the one-dimensional case.

1. INTRODUCTION

In this paper we assume that all spaces are Tychonoff.

Bel'nov [1] proved that every space X can be embedded into a homogeneous space G_X such that $\text{ind } G_X = \text{ind } X$, $\text{Ind } G_X = \text{Ind } X$, and $\dim G_X = \dim X$ in the case when the corresponding dimension of X is finite. The space G_X is a free group over X with continuous right translations. Bel'nov (see [5]) asked whether every space X can be embedded into a topological group G with $\dim G \leq \dim X$.

Shakhmatov [5] proved that if $n \neq 0, 1, 3, 7$, then the n -dimensional sphere S^n cannot be embedded into an n -dimensional topological group. Thus the answer to the above question is no. However, Shakhmatov also proved that in case $\dim X = 0$ the answer to this question is yes. Hence it is natural to ask whether every space X can be embedded into a topological group G with $\dim G \leq \dim X$ in case $\dim X = 1$ or 3 or 7.

In this paper we prove that the answer to this question is no in case $\dim X = 1$. Namely, we prove that if a topological group G contains the bouquet $S^1 \vee S^1$, then $\dim G \geq 2$ holds. Obviously, $\dim(S^1 \vee S^1) = 1$. Hence the bouquet $S^1 \vee S^1$ is a counterexample to a question of Bel'nov in the one-dimensional case.

2. PRELIMINARIES AND LEMMAS

We denote by $\mathbb{R}$, $\mathbb{Q}$, I , and S^1 the real line, the rational space, the closed unit interval $[0, 1]$, and the 1-sphere, respectively. We put $\partial I^2 = (I \times \{0, 1\}) \cup (\{0, 1\} \times I)$. We denote by d the Euclidean metric on $\mathbb{R}^3$. For two continuous

Received by the editors January 16, 1990 and, in revised form, November 5, 1990.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 54F45, 54C25; Secondary 05C10, 22A05.

Key words and phrases. Topological group, dimension, embedding, bouquet, graph.

mappings f and g of a space X into $\mathbb{R}^3$ we put

$$d(f, g) = \sup\{d(f(x), g(x)) : x \in X\}.$$

For a subset X of $\mathbb{R}^3$ and for any $\varepsilon > 0$ we put

$$U(X, \varepsilon) = \{\alpha \in \mathbb{R}^3 : d(\alpha, X) < \varepsilon\}.$$

For a subset X of $\mathbb{R}^3$ we denote by i_X the inclusion mapping of X into $\mathbb{R}^3$. For a mapping $f: X \rightarrow Y$ and for a subset A of X we denote by $f|_A$ the restriction of f to A . Let N_1^3 be the subspace of $\mathbb{R}^3$ consisting of all points which have at most one rational coordinate. The following lemma is needed in §3.

2.1. Lemma. *Let X be a compact metrizable space with $\dim X \leq 1$. Then for every continuous mapping $f: X \rightarrow \mathbb{R}^3$ and for any $\varepsilon > 0$ there exists an embedding $g: X \rightarrow \mathbb{R}^3$ such that $d(f, g) < \varepsilon$ and $g(X) \subset N_1^3$.*

Proof. See the proof of [2, 1.11.5].

Let us set

$$\begin{aligned} L &= \{(x, y, 0) \in \mathbb{R}^3 : x^2 + y^2 = 9\}, \\ M &= \{(x, 0, z) \in \mathbb{R}^3 : (x-2)^2 + z^2 = 1\}, \\ C &= \{(x, y, 0) \in \mathbb{R}^3 : x^2 + y^2 = 4\}, \\ Z &= \{(0, 0, z) \in \mathbb{R}^3 : z \in \mathbb{R}\}, \\ \mathbb{T}^2 &= \{\alpha \in \mathbb{R}^3 : d(\alpha, C) = 1\}. \end{aligned}$$

Then, obviously, $\mathbb{T}^2$ is a torus. For every θ , $0 \leq \theta < 2\pi$, we put

$$F_\theta = \{(r \cos \theta, r \sin \theta, z) \in \mathbb{R}^3 : r > 0 \text{ and } z \in \mathbb{R}\}.$$

Let K be a polyhedral 1-sphere satisfying the following two conditions (*):

$$\begin{aligned} (*) \quad & K \subset U(C, 1) \cap ((\mathbb{R} \times \mathbb{Q}) \cup (\mathbb{Q} \times \mathbb{R})) \times \{0\} \quad \text{and} \\ & |F_\theta \cap K| = 1 \quad \text{for every } \theta, \quad 0 \leq \theta < 2\pi. \end{aligned}$$

We shall construct a continuous mapping $p_K: \mathbb{R}^3 - (K \cup Z) \rightarrow \mathbb{T}^2$. For every θ , $0 \leq \theta < 2\pi$, let $p_\theta: F_\theta - \{x_\theta\} \rightarrow F_\theta \cap \mathbb{T}^2$ be the projection of $F_\theta - \{x_\theta\}$ from the point x_θ onto $F_\theta \cap \mathbb{T}^2$, where $\{x_\theta\} = F_\theta \cap K$. Let $p_K: \mathbb{R}^3 - (K \cup Z) \rightarrow \mathbb{T}^2$ be the mapping defined by

$$p_K(r \cos \theta, r \sin \theta, z) = p_\theta(r \cos \theta, r \sin \theta, z)$$

for every $(r \cos \theta, r \sin \theta, z) \in \mathbb{R}^3 - (K \cup Z)$. Then p_K is continuous. The proof of the following lemma is easy, so we omit the proof.

2.2. Lemma. *For every $\varepsilon > 0$ there exists a polyhedral 1-sphere K with property (*) such that $d(i_{U(\mathbb{T}^2, \varepsilon)}, p_K|_{U(\mathbb{T}^2, \varepsilon)}) < 2\varepsilon$.*

Let $\lambda: I^2 \rightarrow \mathbb{T}^2$ be the natural quotient such that

$$\begin{aligned} \lambda(x, 0) &= \lambda(x, 1) = (3 \cos 2\pi x, 3 \sin 2\pi x, 0), \\ \lambda(0, x) &= \lambda(1, x) = (2 + \cos 2\pi x, 0, \sin 2\pi x), \end{aligned}$$

and

$$\lambda(1/2, 1/2) = (-1, 0, 0).$$

Let $q: I^2 - \{(1/2, 1/2)\} \rightarrow \partial I^2$ be the projection of $I^2 - \{(1/2, 1/2)\}$ from the point $(1/2, 1/2)$ onto ∂I^2 . It is obvious that $|\lambda q \lambda^{-1}(\alpha)| = 1$ for every $\alpha \in \mathbb{T}^2 - \{(-1, 0, 0)\}$. Thus we can define the mapping $\mu: \mathbb{T}^2 - \{(-1, 0, 0)\} \rightarrow \mathbb{T}^2$ such that $\{\mu(\alpha)\} = \lambda q \lambda^{-1}(\alpha)$. Then we have $\mu(\mathbb{T}^2 - \{(-1, 0, 0)\}) = L \cup M$ and μ is continuous. The proof of the following lemma is also easy, so we omit the proof.

2.3. Lemma. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$d(i_{U(L \cup M, \delta) \cap \mathbb{T}^2}, \mu|_{U(L \cup M, \delta) \cap \mathbb{T}^2}, \mu|_{U(L \cup M, \delta) \cap \mathbb{T}^2}) < \varepsilon.$$

3. THE MAIN THEOREM

By Lemma 2.3, we can take $\delta > 0$ such that

$$d(i_{U(L \cup N, \delta) \cap \mathbb{T}^2}, \mu|_{U(L \cup M, \delta) \cap \mathbb{T}^2}) < 1/3.$$

We fix δ in this section.

3.1. Lemma. *Let $f: I^2 \rightarrow \mathbb{R}^3$ be a continuous mapping such that $d(f|_{\partial I^2}, \lambda|_{\partial I^2}) < \delta/4$. Then $\dim f(I^2) \geq 2$ holds.*

Proof. Suppose that $\dim f(I^2) \leq 1$. By Lemma 2.1, we can take an embedding $g: f(I^2) \rightarrow \mathbb{R}^3$ such that $d(i_{f(I^2)}, g) < \delta/4$ and $g \circ f(I^2) \subset N_1^3$. By Lemma 2.2, we can take a polyhedral 1-sphere K with property $(*)$ such that $d(i_{U(\mathbb{T}^2, \delta/2)}, p_K|_{U(\mathbb{T}^2, \delta/2)}) < \delta$. Note that $g \circ f(I^2) \cap (K \cup Z) = \emptyset$, because $g \circ f(I^2) \subset N_1^3$. Since $K \subset ((\mathbb{R} \times \mathbb{Q}) \cup (\mathbb{Q} \times \mathbb{R})) \times \{0\}$ and $g \circ f(I^2) \cap ([-3, 0] \times \{(0, 0)\}) = \emptyset$, we have $(-1, 0, 0) \notin p_K \circ g \circ f(I^2)$. Let $\nu = \mu \circ p_K \circ g \circ f$. Then ν is a continuous mapping of I^2 into $\mathbb{T}^2$ such that $\nu(I^2) \subset L \cup M$ and $d(\nu|_{\partial I^2}, \lambda|_{\partial I^2}) < 1/3$. We take a lifting $\tilde{\nu}: I^2 \rightarrow \mathbb{R}^2$ of ν . Let us set $A_i = I \times \{i\}$ for $i = 0, 1$, and $B_i = \{i\} \times I$ for $i = 0, 1$. Since $d(\nu|_{\partial I^2}, \lambda|_{\partial I^2}) < 1/3$, we may assume that

$$\tilde{\nu}(A_i) \subset V(A_i, 1/3) \quad \text{for } i = 0, 1,$$

and

$$\tilde{\nu}(B_i) \subset V(B_i, 1/3) \quad \text{for } i = 0, 1,$$

where $V(N, \varepsilon)$ is the ε -neighborhood of N in $\mathbb{R}^2$. Let us set $D = \mathbb{R} \times \{1/2\}$ and $E = \{1/2\} \times \mathbb{R}$. Then D is a partition in $\mathbb{R}^2$ between $\tilde{\nu}(A_0)$ and $\tilde{\nu}(A_1)$, and E is a partition in $\mathbb{R}^2$ between $\tilde{\nu}(B_0)$ and $\tilde{\nu}(B_1)$. Thus $\tilde{\nu}^{-1}(D)$ is a partition in I^2 between A_0 and A_1 , and $\tilde{\nu}^{-1}(E)$ is a partition in I^2 between B_0 and B_1 . Thus we have $\tilde{\nu}^{-1}(D) \cap \tilde{\nu}^{-1}(E) \neq \emptyset$. On the other hand, since $\nu(I^2) \subset L \cup M$, we have $\tilde{\nu}(I^2) \subset p^{-1}(L \cup M)$, where $p: \mathbb{R}^2 \rightarrow \mathbb{T}^2$ is the covering mapping. Thus we have $D \cap E \cap \tilde{\nu}(I^2) = \emptyset$. This implies that $\tilde{\nu}^{-1}(D) \cap \tilde{\nu}^{-1}(E) = \emptyset$. This is a contradiction. Hence we have $\dim f(I^2) \geq 2$. Lemma 3.1 has been proved.

Recall that the bouquet $S^1 \vee S^1$ is the one point union of S^1 and S^1 with the common point.

We are now in a position to establish our main theorem.

3.2. Theorem. *If a topological group G contains the bouquet $S^1 \vee S^1$, then $\dim G \geq 2$ holds.*

Proof. Suppose that a topological group G with $\dim G \leq 1$ contains the bouquet $S^1 \vee S^1$. Since $L \cup M$ is homeomorphic to $S^1 \vee S^1$ and since G is

homogeneous, we may assume that $e = (3, 0, 0) \in L \cup M \subset G$, where e is the neutral element of G . Let $\varphi_1: I \rightarrow G$ and $\varphi_2: I \rightarrow G$ be the mappings defined by

$$\varphi_1(x) = (3 \cos 2\pi x, 3 \sin 2\pi x, 0)$$

and

$$\varphi_2(x) = (2 + \cos 2\pi x, 0, \sin 2\pi x)$$

for every $x \in I$. Let $\varphi: I^2 \rightarrow G$ be the mapping defined by $\varphi(x, y) = \varphi_1(x) \cdot \varphi_2(y)$ for every $(x, y) \in I^2$, where $\cdot$ is the group composition of G . Then $\varphi(I^2)$ is compact and metrizable. Since $\varphi_1(0) = \varphi_1(1) = \varphi_2(0) = \varphi_2(1) = (3, 0, 0) = e$, we have $\varphi|_{\partial I^2} = \lambda_{\partial I^2}$. Let $f: \varphi(I^2) \rightarrow \mathbb{R}^3$ be a continuous mapping such that $f|_{\varphi(\partial I^2)} = i_{L \cup M}$. Since $\dim G \leq 1$, we have $\dim \varphi(I^2) \leq 1$. Thus, by Lemma 2.1, we can take an embedding $g: \varphi(I^2) \rightarrow \mathbb{R}^3$ such that $d(f, g) < \delta/4$. Then we have $d(g \circ \varphi|_{\partial I^2}, \lambda|_{\partial I^2}) < \delta/4$. Thus, by Lemma 3.1, we have $\dim \varphi(I^2) = \dim g \circ \varphi(I^2) \geq 2$. This is a contradiction. Theorem 3.2 has been proved.

4. REMARKS AND QUESTIONS

4.1 *Remark.* Vopenka ([7] or see [3, 18–11]) constructed a compact space X such that $\dim X = n < \omega$ and $\text{ind } X = \infty$. Shakhmatov [4] proved that the above space X cannot be embedded into a finite-dimensional, locally pseudo-compact topological group. However, it is unknown whether the above space X can be embedded into a topological group with the same dimension of X .

4.2. *Remark.* In the proof of Theorem 3.2 we get a contradiction to assume that $\dim \varphi(I^2) \leq 1$. Because $\varphi(I^2)$ is compact and metrizable, if $\text{ind } G \leq 1$ or $\text{Ind } G \leq 1$, then we have $\dim \varphi(I^2) \leq 1$. Hence the bouquet $S^1 \vee S^1$ cannot be embedded into a topological group G such that $\text{ind } G \leq 1$ or $\text{Ind } G \leq 1$ or $\dim G \leq 1$.

It is well known that every separable metrizable space X with $\dim X = n$ can be embedded into the $(2n+1)$ -dimensional cube I^{2n+1} . Obviously, I^{2n+1} can be embedded into the $(2n+1)$ -dimensional torus $\mathbb{T}^{2n+1}$. Hence every separable metrizable space X with $\dim X = n$ can be embedded into a $(2n+1)$ -dimensional compact metrizable topological group. It is easy to see that every n -dimensional, locally separable metrizable space can be embedded into a $(2n+1)$ -dimensional metrizable topological group.

4.3. *Question.* Does there exist a mapping $\varphi: \omega \rightarrow \omega$ such that every metrizable space X with $\dim X = n$ can be embedded into a (metrizable) topological group G with $\dim G \leq \varphi(n)$?

Shakhmatov also gave some similar problems in [6].

It is obvious that the bouquet $S^1 \vee S^1$ can be embedded into the 2-dimensional torus $\mathbb{T}^2$. However, it is well known that if $n \geq 8$, then the complete graph K_n cannot be embedded into $\mathbb{T}^2$. Since K_8 contains $S^1 \vee S^1$, K_n cannot be embedded into a topological group G with $\dim G \leq 1$ for each $n \geq 8$. Obviously, K_n can be embedded into the 3-dimensional torus $\mathbb{T}^3$.

4.4. *Question.* Let $n \geq 8$. Can the complete graph K_n be embedded into a topological group G with $\dim G \leq 2$?

A space X is *atriodic* if X does not contain the complete bipartite graph $K_{1,3}$. Obviously, the bouquet $S^1 \vee S^1$ is not atriodic.

4.5. *Question.* Can the complete bipartite graph $K_{1,3}$ be embedded into a topological group G with $\dim G = 1$?

If a graph X is atriodic, then every component of X is homeomorphic to S^1 or I or the singleton. Thus an atriodic graph can be embedded into $S^1 \times \mathbb{Z}_p$, where p is the number of components of X and $\mathbb{Z}_p$ is a group consisting of p points with discrete topology. Hence if the answer to the above question is negative, then a graph X can be embedded into a topological group G with $\dim G = 1$ if and only if X is atriodic.

Added in proof. H. Kato and J. Kulesza, independently, constructed an n -dimensional space X such that every topological group G containing X has $\dim G \geq n + 1$. They also gave a negative answer to Question 4.5.

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