EXTREME DOUBLY STOCHASTIC MEASURES WITH FULL SUPPORT

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ABSTRACT. We give an elementary construction of doubly stochastic measures on $I \times I$ with full support.

1. INTRODUCTION

Let m denote Lebesgue measure on $I = [0, 1]$, the unit interval. A probability measure μ on $I \times I$ is doubly stochastic if for any measurable $A \subseteq I$,

$$\mu(A \times I) = \mu(I \times A) = m(A).$$

An $n \times n$ nonnegative matrix is doubly stochastic if its rows and columns all sum to 1. A doubly stochastic measure (resp. $n \times n$ matrix) is extreme if it is not the positive linear combination of any pair of distinct doubly stochastic measures (resp. $n \times n$ matrices). An extreme measure (resp. $n \times n$ matrix) is just an extreme point in the appropriate convex set. It is well known that the extreme doubly stochastic matrices are exactly the permutation matrices, i.e., the matrices with exactly one nonzero entry in each row and column. In contrast to the sparseness of such matrices, extreme doubly stochastic measures may have full support. We construct such a measure. Other constructions are already known [L]; our goal is simplicity.

2. NOTATION

If $M = (a_{ij})$ is a $2^k \times 2^k$ permutation matrix, let τ_M be the measure preserving transformation on I defined by M . More specifically, when $m_{ij} = 1$,

$$\tau_M(x) = x + (j - i) \cdot 2^{-k} \text{ for } i/2^k \leq x < (i + 1)/2^k.$$

Set $I_m^k = [m2^{-k}, (m + 1)2^{-k}) \subseteq I$, where $0 \leq m < 2^k$. Set $B_{m,n}^k = I_m^k \times I_n^k$. We call these dyadic squares.

As usual, $\delta_{a,b} = 1$ if $a = b$ and $\delta_{a,b} = 0$ if $a \neq b$.

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3. CONSTRUCTIONS WITH PERMUTATION MATRICES

We describe a procedure to build new permutation matrices from old.

Given a $2^k \times 2^k$ permutation matrix M and $h \geq k$, we construct a $2^{k+h} \times 2^{k+h}$ permutation matrix called $\gamma_h(M) = (a_{i,j})$, $0 \leq i, j < 2^{k+h}$.

First, let $M' = (b_{i,j})$, $0 \leq i, j < 2^{k+h}$ be the unique $2^{k+h} \times 2^{k+h}$ permutation matrix with $\tau_{M'} = \tau_M$. (We obtain M' by replacing those entries of M that are 1 by $2^h \times 2^h$ identity matrices and those entries that are 0 by $2^h \times 2^h$ zero matrices.)

Entries of $\gamma_h(M)$ of the form $a_{p \cdot 2^{k+h} + q, p' \cdot 2^{k+h} + q'}$ with $0 \leq p, q, p', q' < 2^k$ are defined equal to $\delta_{p,q'} \delta_{q,p'}$; for the remaining entries, set $a_{i,j} = b_{i,j}$.

Alternatively, we may describe $\tau_{\gamma_h(M)}$. $B = \bigcup_{s=0}^{2^k} I_{s2^{h-k}}^h$ is an invariant set for τ_M . Having fixed h , we will call B the exceptional set for τ_M . Note that its measure, $m(B) = 2^{k-h}$, is small when h is large. For $x \notin B$, we have $\tau_{\gamma_h(M)}(x) = \tau_M(x)$. On the other hand, for $0 \leq s, t < 2^k$, $\tau_{\gamma_h(M)}$ brings $I_{s2^{h-k}}^{k+h}$ to $I_{t2^{h-k}}^{k+h}$ by translation. This makes it clear that $\gamma_h(M)$ is a permutation matrix. Moreover, $m(I_{s2^{h-k}}^h \cap \tau_{\gamma_h(M)}(I_{t2^{h-k}}^h)) = 2^{-h-k}$, so certainly $m(I_s^k \cap \tau_{\gamma_h(M)}(I_t^k)) \geq 2^{-h-k}$.

4. EXTREME DOUBLY STOCHASTIC MEASURES WITH FULL SUPPORT

Let M_0 be any 2×2 permutation matrix, and put $M_{i+1} = \gamma_{3 \cdot 4^i}(M_i)$. Then M_i is a $2^{4^i} \times 2^{4^i}$ permutation matrix. Let $\tau_1, \tau_2, \dots$ be the transformations defined by $M_1, M_2, \dots$, and let $B_1, B_2, \dots$ be the corresponding exceptional sets, as described above.

Then $m(B_i) = 2^{-2 \cdot 4^i}$, hence $\sum_{i=1}^{\infty} m(B_i) < \infty$.

For $x \notin \bigcap_{i=1}^{\infty} \bigcup_{k=i}^{\infty} B_k$, the sequence $(\tau_k(x))$ is finally constant, and it follows that $\tau(x) := \lim \tau_k(x)$ exists almost everywhere. Similarly, we have $m(\tau^{-1}(B)) = \lim m(\tau_k^{-1}(B)) = m(B)$ for any measurable subset B of I .

Put $\tilde{B}_i = \bigcup_{k=i}^{\infty} B_k$. Then $m(\tilde{B}_i) < 2 \cdot 2^{-2 \cdot 4^i} \leq 2^{-4^i}$.

For $0 \leq s, t < 2^{4^i}$, we have $m(\tau_{i+1}^{-1}(I_s^{4^i}) \cap I_t^{4^i}) \geq 2^{-4 \cdot 4^i} = 2^{-4^{i+1}}$. Since $\tau^{-1}(I_s^{4^i}) \cap I_t^{4^i} \supseteq \tau_{i+1}^{-1}(I_s^{4^i}) \cap I_t^{4^i} \setminus \tilde{B}_{i+1}$, it follows that $m(\tau^{-1}(I_s^{4^i}) \cap I_t^{4^i}) > 0$, i.e., the image of Lebesgue measure on the graph of τ , call it β_τ , has full support.

It is well known that the image of Lebesgue measure on the graph of a measure-preserving transformation is an extreme doubly stochastic measure, so the example is complete.

In order that the exposition be entirely self-contained and because the method itself may have some interest, the remainder of this paper develops an alternative approach to verifying extremality.

5. A SUFFICIENT CONDITION FOR EXTREMALITY

Let μ , μ_1 , and μ_2 be doubly stochastic measures where $\mu = \alpha\mu_1 + (1-\alpha)\mu_2$, $\mu_1 \neq \mu_2$, $0 < \alpha < 1$. Interchange the roles of μ_1 and μ_2 , if necessary, so that we may take $\alpha \in (0, \frac{1}{2}]$. Being distinct, μ_1 and μ_2 cannot agree on all dyadic squares, so

$$(1) \quad |\mu_1(B_{m_0, n_0}^k) - \mu_2(B_{m_0, n_0}^k)| = \varepsilon > 0$$

for some B_{m_0, n_0}^k . Any doubly stochastic measure assigns B_{m_0, n_0}^k a measure in the interval $[0, 2^{-k}]$, since B_{m_0, n_0}^k sits inside a horizontal (and a vertical) strip of measure 2^{-k} . Then

$$(2) \quad \mu(B_{m_0, n_0}^k) \in [\alpha\varepsilon, 2^{-k} - \alpha\varepsilon].$$

To see this, consider the interval $J \subset [0, 2^{-k}]$ whose endpoints are $\mu_1(B_{m_0, n_0}^k)$ and $\mu_2(B_{m_0, n_0}^k)$. J has length ε . The distance from

$$(3) \quad \mu(B_{m_0, n_0}^k) = \alpha\mu_1(B_{m_0, n_0}^k) + (1-\alpha)\mu_2(B_{m_0, n_0}^k)$$

to the nearest endpoint of J is precisely $\alpha\varepsilon$, since $\alpha \leq \frac{1}{2}$. So, a fortiori, it cannot be closer than $\alpha\varepsilon$ to the endpoints of $[0, 2^{-k}]$.

Now there are four $(k+1)$ -order dyadic squares contained in B_{m_0, n_0}^k . For at least one, call it B_{m_1, n_1}^{k+1} , we must have

$$(4) \quad |\mu_1(B_{m_1, n_1}^{k+1}) - \mu_2(B_{m_1, n_1}^{k+1})| \geq \varepsilon/4$$

by (1). Arguing as we did for (2) shows

$$(5) \quad \mu(B_{m_1, n_1}^{k+1}) \in [\alpha\varepsilon/4, 2^{-(k+1)} - \alpha\varepsilon/4]$$

and continuing in this fashion yields a nested sequence

$$(6) \quad B_{m_0, n_0}^k \supset B_{m_1, n_1}^{k+1} \supset \dots \supset B_{m_j, n_j}^{k+j} \supset \dots$$

of dyadic squares such that

$$(7) \quad \mu(B_{m_j, n_j}^{k+j}) \in [\alpha\varepsilon 4^{-j}, 2^{-(k+j)} - \alpha\varepsilon 4^{-j}] \quad \text{for } j = 0, 1, \dots$$

We conclude that a sufficient condition for μ to be an extreme doubly stochastic measure is that there be no nested sequence as in (6) that satisfies (7) for any $\varepsilon, \alpha > 0$. Strengthening (7) slightly produces a simpler sufficient condition that does not involve k, ε , and α and suffices for our purposes. We shall state it as a lemma.

Lemma 1. *Let μ be a doubly stochastic measure on $I \times I$, and fix constants κ , $\delta > 0$. Assume that on any nested sequence of dyadic squares,*

$$(8) \quad B_{0,0}^0 \supset B_{m_1,n_1}^1 \supset \cdots \supset B_{m_l,n_l}^l \supset \cdots,$$

the following condition fails for infinitely many l :

$$(9) \quad \mu(B_{m_l,n_l}^l) \in [\kappa 4^{-(1+\delta)l}, 2^{-l} - \kappa 4^{-(1+\delta)l}].$$

Then μ is extreme.

6. APPLYING THE LEMMA

Observe that if M is a $2^k \times 2^k$ matrix and $g \leq k$, we have

$$\beta_M(B_{s,t}^g) = m(I_s^g \cap \tau_M(I_t^g)).$$

From this we may deduce

$$|\beta_{\gamma_h(M)}(B_{s,t}^g) - \beta_M(B_{s,t}^g)| \leq m((I_s^g \cap \tau_M(I_t^g)) \Delta (I_s^g \cap \tau_{\gamma_h(M)}(I_t^g))) \leq 2^{k-g-h},$$

since I_t is the union of 2^{k-g} intervals of length 2^{-k} and τ_M and $\tau_{\gamma_h(M)}$ agree on each interval outside a subinterval of length $2^{-k} 2^h / 2^k$.

In particular, we have

$$|\beta_{M_{i+j}}(B_{s,t}^{4^i}) - \beta_{M_{i+j+1}}(B_{s,t}^{4^i})| \leq 2^{-(2 \cdot 4^j + 1)4^i}.$$

Using symmetric differences again shows that

$$|\beta\tau(B_{s,t}^{4^i}) - \beta_{M_i}(B_{s,t}^{4^i})| \leq \sum_{j=1}^{\infty} \leq 2^{-(2 \cdot 4^j + 1)4^i} < 2 \cdot 2^{-3 \cdot 4^i}.$$

Since $\beta_{M_i}(B_{s,t}^{4^i})$ is either 0 or 4^{-i} , we may apply the lemma.

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