LATTICES WITH THE ALEXANDROV PROPERTIES

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(Communicated by Andrew M. Bruckner)

ABSTRACT. By an Alexandrov lattice we mean a δ normal lattice $\mathcal{L}$ of subsets of an abstract set X , such that the set of $\mathcal{L}$ -regular countably additive bounded measures, denoted by $\text{MR}(\sigma, \mathcal{L})$, is sequentially closed in the set of $\mathcal{L}$ -regular finitely additive bounded measures on the algebra generated by $\mathcal{L}$, i.e., if $\mu_n \in \text{MR}(\sigma, \mathcal{L})$ and $\mu_n \rightarrow \mu$ (weakly) then $\mu \in \text{MR}(\sigma, \mathcal{L})$.

For a pair of lattices $\mathcal{L}_1 \subset \mathcal{L}_2$ in X sufficient conditions are indicated to determine when $\mathcal{L}_1$ Alexandrov implies that $\mathcal{L}_2$ is also Alexandrov and vice versa. The extension of this situation is given where $T: X \rightarrow Y$, $\mathcal{L}_1$ and $\mathcal{L}_2$ are lattices of subsets of X of Y respectively, and T is $\mathcal{L}_1 - \mathcal{L}_2$ continuous.

1. INTRODUCTION

Alexandrov's fundamental theorem [1] states that if $\mathcal{L}$ is a δ normal lattice of subsets of a set X and if $\mathcal{L}$ is also complement generated (completely normal), then $\mu_n \in \text{MR}(\sigma, \mathcal{L})$ and $\mu_n \xrightarrow{w} \mu$, where $\mu \in \text{MR}(\mathcal{L})$, imply $\mu \in \text{MR}(\sigma, \mathcal{L})$ (see below for definitions). A different proof of this appears in [4] as well as in [6] where it is proved for the case of the lattice of zero sets of a completely regular topological space.

Our aim here is to consider other lattices that have this fundamental Alexandrov property; such lattices will be called Alexandrov lattices. In particular, we will consider the situation of two lattices $\mathcal{L}_1, \mathcal{L}_2$ of subsets of X such that $\mathcal{L}_1 \subset \mathcal{L}_2$ and will determine conditions under which $\mathcal{L}_1$ is Alexandrov implies $\mathcal{L}_2$ is Alexandrov and conversely. We will also consider the case of two sets X and Y and appropriate lattice mappings between them and subsequently adjoint mappings in order to determine further Alexandrov lattices. Our terminology and notation are consistent with [1, 2]. For the reader's convenience, we will repeat several special notations and the lattice terminology needed in this paper.

2. DEFINITIONS AND NOTATIONS

Let X be a set and $\mathcal{L}$ be any lattice of subsets of X . We shall always assume that $\phi, X \in \mathcal{L}$. We use the following notation: R for the real numbers, x for the general element of X , $\mathfrak{A}(\mathcal{L})$ for the smallest algebra containing $\mathcal{L}$, $\sigma(\mathcal{L})$ for the smallest σ -algebra containing $\mathcal{L}$, $\delta(\mathcal{L})$ for the smallest

Received by the editors October 29, 1990.

1991 *Mathematics Subject Classification*. Primary 28A32, 46E27, 28C15, 46G12.

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 0002-9939/92 \$1.00 + \$.25 per page

δ -lattice containing $\mathcal{L}$, $\tau(\mathcal{L})$ for the smallest τ -lattice containing $\mathcal{L}$, and $s(\mathcal{L})$ for the Souslin sets determined by $\mathcal{L}$. $\delta(\mathcal{L})$ is the set of all $\bigcap L_i$ with $L_i \in \mathcal{L}$, $i = 1, 2, \dots$. If $\mathcal{L}$ is closed under countable intersections, then $\mathcal{L}$ is δ or a δ -lattice. $\mathcal{L}$ is *complemented* if $A \in \mathcal{L}$ implies $A' \in \mathcal{L}$ where $A' = X - A$. $\mathcal{L}'$ is the class of all complements of $\mathcal{L}$ -sets, i.e., $\mathcal{L}' = \{L' : L \in \mathcal{L}\}$. $\mathcal{L}$ is *complement generated* if $A \in \mathcal{L}$ implies $A = \bigcap A'_i$ where $A_i \in \mathcal{L}$, $i = 1, 2, \dots$.

$\mathcal{L}$ is *disjunctive* if for any $A \in \mathcal{L}$ and $x \notin A$ there exists a $B \in \mathcal{L}$ such that $x \in B$ and $A \cap B = \emptyset$. $\mathcal{L}$ is *countably paracompact* (c.p.) if $A_n \in \mathcal{L}$ and $A_n \downarrow \phi$ imply there exist $B_n \in \mathcal{L}$ such that $A_n \subset B'_n$ and $B'_n \downarrow \phi$.

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two lattices of subsets of X and $\mathcal{L}_1 \subset \mathcal{L}_2$. $\mathcal{L}_1$ *seminseparates* (s.s.) $\mathcal{L}_2$ if for every $A \in \mathcal{L}_1$ and $B \in \mathcal{L}_2$, $A \cap B = \emptyset$ implies there exists a $C \in \mathcal{L}_1$ such that $B \subset C$ and $A \cap C = \emptyset$; $\mathcal{L}_1$ *separates* $\mathcal{L}_2$ if for all $A, B \in \mathcal{L}_2$, $A \cap B = \emptyset$ implies there exist $C, D \in \mathcal{L}_1$ such that $A \subset C$, $B \subset D$, $C \cap D = \emptyset$. $\mathcal{L}_2$ is $\mathcal{L}_1$ *countably paracompact* if for every $A_n \in \mathcal{L}_2$, $A_n \downarrow \phi$ imply there exist $B_n \in \mathcal{L}_1$ such that $A_n \subset B'_n$ and $B'_n \downarrow \phi$. $\mathcal{L}_2$ is $\mathcal{L}_1$ *countably bounded* (c.b.) if for all $A_n \in \mathcal{L}_2$, $A_n \downarrow \phi$ imply there exist $B_n \in \mathcal{L}_1$ such that $A_n \subset B_n$ and $B_n \downarrow \phi$.

A function $f: X \rightarrow R \cup \{\pm\infty\}$ is $\mathcal{L}_1$ -*continuous* if $f^{-1}(C) \in \mathcal{L}$ for every closed set $C \subset R \cup \{\pm\infty\}$. The algebra of all $\mathcal{L}$ continuous function is denoted by $C(\mathcal{L})$; the algebra of bounded continuous functions is denoted by $C_b(\mathcal{L})$. The set whose general element is a zero set of $\mathcal{L}$ is denoted by $\mathcal{Z}(\mathcal{L})$; $Z \in \mathcal{Z}(\mathcal{L})$ iff $Z = f^{-1}(0)$ for some $\mathcal{L}$ -continuous function f . A *measure* on $\mathcal{A}(\mathcal{L})$ is a finitely additive bounded real-valued set function. $M(\mathcal{L})$ denotes the set of all measures on $\mathcal{A}(\mathcal{L})$. A measure μ is said to be σ -smooth on $\mathcal{L}$ if for every $A_n \in \mathcal{L}$, $A_n \downarrow \phi$ imply $\mu(A_n) \rightarrow 0$. A measure $\mu \in M(\mathcal{L})$ is said to be $\mathcal{L}$ -*regular* if for every $A \in \mathcal{A}(\mathcal{L})$ and every $\varepsilon > 0$, there exists an $L \in \mathcal{L}$ such that $L \subset A$ and $|\mu(A) - \mu(L)| < \varepsilon$. The set whose general element is an element of $\mathcal{L}$, which is σ -smooth on $\mathcal{L}$, is denoted by $M(\sigma, \mathcal{L})$. Moreover, we use the notation $MR(\sigma, \mathcal{L}) = MR(\mathcal{L}) \cap M(\sigma, \mathcal{L})$. The set of all measures such that $\mu(A) = \{0, 1\}$ for every $A \in \mathcal{A}(\mathcal{L})$ and $\mu(X) = 1$ is denoted by $I(\mathcal{L})$. The set of all $\{0, 1\}$ -valued $\mathcal{L}$ -regular measures is denoted by $IR(\mathcal{L})$, i.e., $IR(\mathcal{L}) = I(\mathcal{L}) \cap MR(\mathcal{L})$.

For the restriction of $\nu \in MR(\mathcal{L}_2)$ to $\mathcal{A}(\mathcal{L}_1)$, we adopt the notation $\nu|_{\mathcal{L}_1}$, or simply, $\nu|$. Note that if $\nu \in MR(\mathcal{L}_2)$ and if $\mathcal{L}_1$ s.s. $\mathcal{L}_2$, then $\nu| \in MR(\mathcal{L}_1)$.

We conclude this section with the following general extension theorem and state for reference Alexandrov's Representation Theorem (A.R.T).

Theorem 2.1 [4]. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$, $\mathcal{L}_1 \subset \mathcal{L}_2$, be two lattices of subsets of X . Then, any measure $\mu \in MR(\mathcal{L}_1)$ can be extended to a ν , $\nu \in MR(\mathcal{L}_2)$, and if $\mathcal{L}_1$ separates $\mathcal{L}_2$ then ν is unique. If $\mu \in MR(\sigma, \mathcal{L}_1)$ and $\mathcal{L}_2$ is $\mathcal{L}_1$ countably paracompact or countably bounded, then $\nu \in MR(\sigma, \mathcal{L}_2)$.*

Theorem (Alexandrov) [1b]. *Let $\mathcal{L}$ be a δ normal lattice. Then the conjugate space $C_b(\mathcal{L})'$ of $C_b(\mathcal{L})$ is $MR(\mathcal{L})$. In more details: To every bounded linear functional Φ there corresponds a unique $\mu \in MR(\mathcal{L})$ such that $\Phi(f) = \int f d\mu$ with $\|\Phi\| = |\mu|$. The positive and negative parts of Φ correspond to those of μ . Furthermore, if Φ is nonnegative, then $\forall A \in \mathcal{L}$, $\mu(A) = \inf \Phi(f)$ where $\inf$ is taken over all f in $C_b(\mathcal{L})$ such that $\chi_A \leq f \leq 1$, where χ_A is the characteristic function of A .*

3. ALEXANDROV LATTICES

We can topologize $\text{MR}(\mathcal{L})$ with the weak $(*)$ topology as follows: If $\mu_\alpha \in \text{MR}(\mathcal{L})$ then μ_α converges to μ in the weak topology if and only if $\int f d\mu_\alpha$ converges to $\int f d\mu$, $\forall f \in C_b(\mathcal{L})$. In other words, we write

$$\mu_\alpha \in \text{MR}(\mathcal{L}), \quad \mu_\alpha \xrightarrow{w} \mu \Leftrightarrow \int f d\mu_\alpha \rightarrow \int f d\mu \quad \text{for all } f \in C_b(\mathcal{L}).$$

Let $\mathcal{L}$ be δ normal and complement generated (completely normal) lattice of subsets of X . Then, Alexandrov's Fundamental Theorem [1b, 4] states that $\text{MR}(\sigma, \mathcal{L})$ is weakly sequentially closed in $\text{MR}(\mathcal{L})$, i.e., if $\mu_n \in \text{MR}(\sigma, \mathcal{L})$ and $\mu_n \xrightarrow{w} \mu$, then $\mu \in \text{MR}(\sigma, \mathcal{L})$.

Definition. A δ normal lattice $\mathcal{L}$ of subsets of X is said to be an *Alexandrov lattice* if $\mu_n \in \text{MR}(\sigma, \mathcal{L})$ and $\mu_n \xrightarrow{w} \mu$, where $\mu \in \text{MR}(\mathcal{L})$, imply $\mu \in \text{MR}(\sigma, \mathcal{L})$.

Proposition 3.1. Let $\mathcal{L}_1$ and $\mathcal{L}_2$, $\mathcal{L}_1 \subset \mathcal{L}_2$, be δ normal lattices. If $\mathcal{L}_2$ is $\mathcal{L}_1$ c.p. or c.b. and $\mathcal{L}_1$ s.s. $\mathcal{L}_2$, then $\mathcal{L}_1$ Alexandrov implies that $\mathcal{L}_2$ is Alexandrov.

Proof. Suppose $\nu_n \in \text{MR}(\sigma, \mathcal{L}_2)$ and $\nu_n \xrightarrow{w} \nu$, $\nu \in \text{MR}(\mathcal{L}_2)$. Since $\mathcal{L}_1$ s.s. $\mathcal{L}_2$ and $C_b(\mathcal{L}_2) \supset C_b(\mathcal{L}_1)$, then we have

$$\mu_n \xrightarrow{w} \mu, \quad \mu_n = \nu_n| \in \text{MR}(\sigma, \mathcal{L}_1), \quad \mu = \nu|.$$

Since $\mathcal{L}_1$ is Alexandrov, $\mu \in \text{MR}(\sigma, \mathcal{L}_1)$ and, consequently, $\nu \in \text{MR}(\sigma, \mathcal{L}_2)$ (since $\mathcal{L}_2$ is $\mathcal{L}_1$ c.p. or c.b.). Thus $\mathcal{L}_2$ is also an Alexandrov lattice.

Remark. If instead of $\mathcal{L}_1$ s.s. $\mathcal{L}_2$ we assume that $\mathcal{L}_1$ is δ and $\sigma(\mathcal{L}_1) \subset s(\mathcal{L}_1)$, then in this case $\mu_n = \nu_n|$ is countable, additive, and, therefore, by Choquet's capacity theorem [5], $\mu_n \in \text{MR}(\sigma, \mathcal{L}_1)$. Also, $\mu_n \xrightarrow{w} \mu = \nu| \in M(\mathcal{L}_1)$, but $\mu \leq \rho$ on $\mathcal{L}_1$, where $\rho \in \text{MR}(\mathcal{L}_1)$ and $\mu(X) = \rho(X)$, and since $\int f d\mu = \int f d\rho$ for all $f \in C_b(\mathcal{L}_1)$, $\mu_n \xrightarrow{w} \rho$. Hence $\rho \in \text{MR}(\sigma, \mathcal{L}_1)$ since $\mathcal{L}_1$ is Alexandrov and consequently $\mu \in M(\sigma, \mathcal{L}_1)$, and if $\mathcal{L}_2$ is $\mathcal{L}_1$ c.b. then $\nu \in \text{MR}(\sigma, \mathcal{L}_2)$ and $\mathcal{L}_2$ is Alexandrov.

Note that if $\mathcal{L}_1$ and $\mathcal{L}_2$ are δ normal and $C(\mathcal{L}_1) = C(\mathcal{L}_2)$, which implies that $\mathcal{L}_1$ separates $\mathcal{L}_2$, and if $\mathcal{L}_2$ is c.p., then $\mathcal{L}_2$ is $\mathcal{L}_1$ c.p. and $\mu \in \text{MR}(\sigma, \mathcal{L}_1)$. Then by Theorem 2.1 μ extends uniquely to $\nu \in \text{MR}(\sigma, \mathcal{L}_2)$. In other words, we have

Corollary 3.1. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be δ normal, $C(\mathcal{L}_2) = C(\mathcal{L}_1)$, and $\mathcal{L}_2$ be c.p. Then $\mathcal{L}_1$ Alexandrov implies that $\mathcal{L}_2$ is Alexandrov.

By Proposition 3.1, we also have

Corollary 3.2. If $\mathcal{L}$ is δ normal and c.p., then $\mathcal{L}$ is $\mathcal{Z}(\mathcal{L})$ c.p. and $\mathcal{L}$ is also Alexandrov since $\mathcal{Z}(\mathcal{L})$ is Alexandrov.

Suppose $\mathcal{L}_1$ and $\mathcal{L}_2$ and δ normal lattices, $\mathcal{L}_1 \subset \mathcal{L}_2$, and $C(\mathcal{L}_1) = C(\mathcal{L}_2)$. Let $\mu_n \xrightarrow{w} \mu$ where $\mu_n \in \text{MR}(\sigma, \mathcal{L}_1)$ and $\mu \in \text{MR}(\mathcal{L}_1)$. Then, if $\nu_n \in \text{MR}(\sigma, \mathcal{L}_2)$ is the unique extension of μ_n , and ν is that of μ , we have $\nu_n \xrightarrow{w} \nu$ and $\nu \in \text{MR}(\sigma, \mathcal{L}_2)$ assuming $\mathcal{L}_2$ is Alexandrov. Therefore, $\mu \in \text{MR}(\sigma, \mathcal{L}_1)$ and $\mathcal{L}_1$ is Alexandrov. This fact together with Corollary 3.1 gives

Corollary 3.3. *If $\mathcal{L}_1$ and $\mathcal{L}_2$ are δ normal lattices, $\mathcal{L}_2$ is c.p. or c.b., and $C(\mathcal{L}_2) = C(\mathcal{L}_1)$, then $\mathcal{L}_1$ is Alexandrov if and only if $\mathcal{L}_2$ is Alexandrov.*

Proposition 3.2. *If $\mathcal{L}$ is a δ normal and c.p. lattice of subsets of X and $\mu_n \xrightarrow{w} \mu$ where $\mu_n \in M(\sigma, \mathcal{L})$, $\mu \in M(\mathcal{L})$, then $\mu \in M(\sigma, \mathcal{L})$.*

Proof. Let $\mu_n \in M(\sigma, \mathcal{L})$ and $\mu_n \xrightarrow{w} \mu$. The functional $\Phi_n(f) = \int f d\mu_n$ is a bounded linear functional of $f \in C_b(\mathcal{L})$ and, by A.R.T., we have $\Phi_n(f) = \int f d\mu_n = \int f d\nu_n$, $\nu_n \in \text{MR}(\mathcal{L})$, $\mu_n(X) = \nu_n(X)$, and $\mu_n \leq \nu_n$ on $\mathcal{L}$. Since $\mathcal{L}$ is normal and c.p., we also have $\nu_n \in \text{MR}(\sigma, \mathcal{L})$. Thus $\int f d\nu_n = \int f d\mu_n \rightarrow \int f d\mu$. Also, by A.R.T., $\Phi(f) = \int f d\mu = \int f d\nu$, $\nu \in \text{MR}(\mathcal{L})$ and $\mu(X) = \nu(X)$ on $\mathcal{L}$. Therefore, $\int f d\nu_n \rightarrow \int f d\nu$ or $\nu_n \xrightarrow{w} \nu$. Since $\mathcal{L}$ is δ normal and c.p., by Corollary 3.2, $\nu \in \text{MR}(\sigma, \mathcal{L})$. On the other hand, we have $\mu \leq \nu$ on $\mathcal{L}$. Therefore, $\mu \in \text{MR}(\sigma, \mathcal{L})$.

Proposition 3.3. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be lattices of subsets of X . Suppose $\mathcal{L}_1$ separates $\mathcal{L}_2$ and $\mathcal{L}_1$ is an Alexandrov lattice. If $M(\sigma, \mathcal{L}_2') \cap \text{MR}(\mathcal{L}_2) \subset M(\sigma, \mathcal{L}_2)$ then $\mathcal{L}_2$ is also an Alexandrov lattice.*

Proof. Let $\nu_n \xrightarrow{w} \nu$ where $\nu_n \in \text{MR}(\sigma, \mathcal{L}_2)$ and $\nu \in \text{MR}(\mathcal{L}_2)$. Since $\mathcal{L}_1$ is an Alexandrov lattice and $\nu_n|_{\mathcal{L}_1} = \mu_n \in \text{MR}(\sigma, \mathcal{L}_1)$, we have $\nu_n|_{\mathcal{L}_1} = \mu_n \xrightarrow{w} \nu|_{\mathcal{L}_1} = \mu \in \text{MR}(\sigma, \mathcal{L}_1)$. $\mathcal{L}_1$ actually coseparates $\mathcal{L}_2$ since $\mathcal{L}_1$ separates $\mathcal{L}_2$ and $\mathcal{L}_1$ is normal. It is not difficult to see that $\nu \in \text{MR}(\sigma, \mathcal{L}_2)$ since ν must be $\mathcal{L}$ -regular on $\mathcal{L}_2'$; namely, $\nu \in \text{MR}(\mathcal{L}_2)$ by hypothesis and $\nu \in M(\sigma, \mathcal{L}_2')$ since ν is $\mathcal{L}_1$ -regular on $\mathcal{L}_2'$. Clearly, if $M(\sigma, \mathcal{L}_2') \cap \text{MR}(\mathcal{L}_2) \subset M(\sigma, \mathcal{L}_2)$ then $\nu \in \text{MR}(\sigma, \mathcal{L}_2)$. Therefore, $\mathcal{L}_2$ is an Alexandrov lattice.

Let $\mathcal{L}_1$ be a lattice of subsets of X and $\mathcal{L}_2$ be a lattice of subsets of Y . Again, we assume that $\mathcal{L}_1$ and $\mathcal{L}_2$ are δ normal.

Let $T: X \rightarrow Y$ be $\mathcal{L}_1$ – $\mathcal{L}_2$ continuous. Consider a mapping $A: C_b(\mathcal{L}_2) \rightarrow C_b(\mathcal{L}_1)$ such that A is linear and bounded.

If the mapping A is defined by $Ag = gT$ where $g \in C_b(\mathcal{L}_2)$, then define the adjoint map by $A': C_b(\mathcal{L}_1)' \rightarrow C_b(\mathcal{L}_2)'$ where $C_b(\mathcal{L}_i)'$ is congruent to $\text{MR}(\mathcal{L}_i)$ ($i = 1, 2$) and $(A'\Phi)(g) = \Phi(Ag)$. By A.R.T. we have $\Phi \leftrightarrow \mu$, $\mu \in \text{MR}(\mathcal{L}_1)$ and $A'\Phi \leftrightarrow \nu$, $\nu \in \text{MR}(\mathcal{L}_2)$. Then, $\Phi(Ag) = \int Ag d\mu$ and $(A'\Phi)(g) = \int g d\nu$ for all $g \in C_b(\mathcal{L}_2)$ and consequently $A': \text{MR}(\mathcal{L}_1) \rightarrow \text{MR}(\mathcal{L}_2)$ where $A'\mu = \nu$ and $\int_Y g d\nu = (A'\Phi)(g) = \Phi(Ag) = \int_X Ag d\mu = \int_X gT d\mu = \int_Y g d\mu T^{-1}$, $g \in C_b(\mathcal{L}_2)$.

Note that A is a linear mapping and that $Ag_1g_2 = g_1Tg_2T = Ag_1Ag_2$. Therefore, A is an algebra homomorphism. Also, we have $\|Ag\| = \|gT\| \leq \|g\|$. Indeed, A is bounded. If T is surjective then $\|Ag\| = \|g\|$, i.e., A is an isometry, and consequently A is invertible.

Some basic properties of A' are collected in the following.

Proposition 3.4. (a) *If $\mu \geq 0$ then $\nu = A'\mu \geq 0$.*

(b) *$A'\mu = \nu \geq \mu T^{-1}$ on $\mathcal{L}_2$ and $\nu(Y) = \mu T^{-1}(Y)$.*

(c) *$A'(IR(\mathcal{L}_1)) \subset IR(\mathcal{L}_2)$.*

(d) *$A'|_{IR(\mathcal{L}_1)}$ is weakly continuous.*

Proof. We show only (b). Further details can be found in [2]. We have $\mu T^{-1}(L) = \int_L d\mu T^{-1} = \int_Y \chi_L d\mu T^{-1} \leq \int_Y g d\mu T^{-1} = \int_Y g d\nu$, where $g \in$

$C_b(\mathcal{L}_2)$ and $\chi_L \leq g \leq 1$. Therefore, $\mu T^{-1}(L) \leq \nu(L)$ for all $L \in \mathcal{L}_2$. If $g = 1$, we obtain $\int_Y d\nu = \int_Y d\mu T^{-1}$. Hence $\nu(Y) = \mu T^{-1}(Y)$.

Proposition 3.5. (a) If $\mathcal{L}_2$ is c.p., then $A'(\text{MR}(\sigma, \mathcal{L}_1)) \subset \text{MR}(\sigma, \mathcal{L}_2)$.

(b) If T is surjective and $\mathcal{L}_1$ is $T^{-1}(\mathcal{L}_2)$ c.b., then $\text{MR}(\sigma, \mathcal{L}_2) \subset A'(\text{MR}(\sigma, \mathcal{L}_1))$.

(c) If (a) and (b) hold then, $A'(\text{MR}(\sigma, \mathcal{L}_1)) = \text{MR}(\sigma, \mathcal{L}_2)$.

Proof. Here we show only (a). Suppose $\mathcal{L}_2$ is c.p. Let $\mu \in \text{MR}(\sigma, \mathcal{L}_1)$ and consider any element of $A'(\text{MR}(\sigma, \mathcal{L}_1))$, $A'\mu$. We must show that $A'\mu = \nu \in \text{MR}(\sigma, \mathcal{L}_2)$. By A.R.T. we have $\mu \leftrightarrow \Phi$ and Φ is σ -smooth. In fact, consider $\{g_n\}$, $g_n \in C_b(\mathcal{L}_2)$, $g_n \downarrow 0$. Then $g_n T \downarrow 0$ and, therefore, $\lim \int g_n T d\mu = 0$. However, $\lim \int g_n T d\mu = \lim \int g_n d\nu = 0$, which means $\Phi \leftrightarrow \nu$ where $\Phi(g) = \int g d\nu$ for all $g \in C_b(\mathcal{L}_2)$. Since Φ is σ -smooth and $\mathcal{L}_2$ is c.p., we have $A'\mu = \nu \in \text{MR}(\sigma, \mathcal{L}_2)$. Hence $A'(\text{MR}(\sigma, \mathcal{L}_1)) \subset \text{MR}(\sigma, \mathcal{L}_2)$.

Proposition 3.6. (1) Under the assumption Proposition 3.5 (a), if $\mathcal{L}_1$ is an Alexandrov lattice and

$$\mu_n \xrightarrow{w} \mu, \quad \mu_n \in \text{MR}(\sigma, \mathcal{L}_1),$$

then $A'\mu_n \xrightarrow{w} A'\mu$ and $A'\mu_n, A'\mu \in \text{MR}(\sigma, \mathcal{L}_2)$.

(2) Under the assumptions Proposition 3.5 (a), (b) and if A is surjective, then $\mathcal{L}_1$ Alexandrov implies that $\mathcal{L}_2$ is Alexandrov.

Proof. (1) Since $\mathcal{L}_1$ is Alexandrov, we have $\mu_n \xrightarrow{w} \mu$, $\mu \in \text{MR}(\sigma, \mathcal{L}_1)$. Then, by Proposition 3.4(d), $A'\mu_n \rightarrow A'\mu$. By Proposition 3.5(a), $A'\mu_n \in \text{MR}(\sigma, \mathcal{L}_2)$ and $A'\mu \in A'(\text{MR}(\sigma, \mathcal{L}_1)) \subset \text{MR}(\sigma, \mathcal{L}_2)$.

(2) Let $\nu_n \in \text{MR}(\sigma, \mathcal{L}_2)$. Then $\nu_n \xrightarrow{w} \nu \in \text{MR}(\mathcal{L}_2)$. By Proposition 3.5(b), we have $\nu_n = A'\mu_n$, $\mu_n \in \text{MR}(\sigma, \mathcal{L}_1)$ and $\nu = A'\mu$, $\mu \in \text{MR}(\mathcal{L}_1)$. Since A is surjective, we have $\int Ag d\mu_n = \int g d\nu_n \rightarrow \int g d\nu = \int Ag d\mu$. Hence $\mu_n \xrightarrow{w} \mu$. Since $\mathcal{L}_1$ is Alexandrov, $\mu \in \text{MR}(\sigma, \mathcal{L}_1)$. Therefore, by Proposition 3.5(a), $A'\mu = \nu \in \text{MR}(\sigma, \mathcal{L}_2)$. Hence $\mathcal{L}_2$ is Alexandrov.

Thus, under the above assumptions, the measure defined on Alexandrov lattices is invariant under adjoint mappings.

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