

AN APPROXIMATION CONNECTED WITH THE EXPONENTIAL FUNCTION

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ABSTRACT. Some recent techniques in the uniform asymptotic expansions of integrals are used to obtain an expansion for a function related to the exponential function. This function is associated with Ramanujan, Watson, Copson, and Buckholtz. The results obtained complement those given by Buckholtz.

1. INTRODUCTION

Let

$$(1.1) \quad \frac{(nz)^n}{n!} S_n(z) = e^{nz} - \sum_{k=0}^n \frac{(nz)^k}{k!}.$$

In 1963 Buckholtz [3] gave an asymptotic expansion of $S_n(z)$ that is valid in the whole complex z -plane except $z = 1$. He proved that if $z \neq 1$, then

$$(1.2) \quad S_n(z) \sim n! \left(\frac{e^z}{nz} \right)^n \phi_K(z) + \sum_{k=0}^{\infty} n^{-k} u_k(z),$$

where

$$(1.3) \quad u_k(z) = \left(\frac{z}{z-1} \frac{d}{dz} \right)^k \frac{z}{1-z} = \frac{(-1)^{k+1} Q_k(z)}{(z-1)^{2k+1}},$$

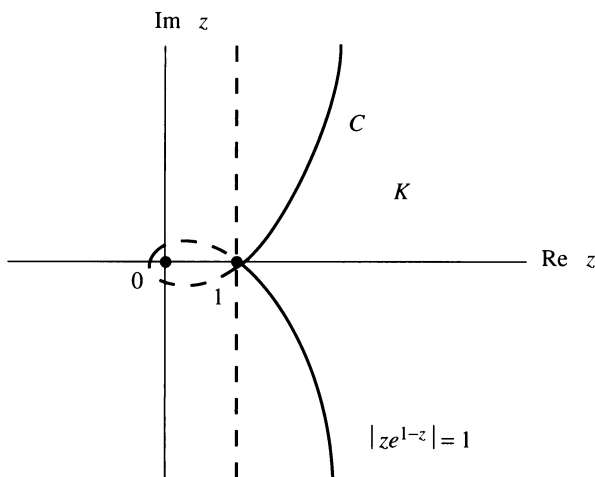
$Q_k(z)$ is a polynomial of degree k with positive integer coefficients, and $\phi_K(z)$ is the characteristic function of the set $K = \{z/|z| \geq 1 \text{ and } |ze^{1-z}| \leq 1\}$, see Figure 1. Furthermore, Buckholtz proved that this expansion holds uniformly in the region where $d_K(z) \geq \varepsilon > 0$, $d_K(z)$ being the distance from z to the set K . In particular, for $z = -1$, it yields result of Copson [4]. As $z \rightarrow 1$, each of the coefficients $u_k(z)$ in (1.2) become unbounded. On the other hand,

$$(1.4) \quad S_n(1) = \frac{n!}{2} \left(\frac{2}{3} \right)^n - \frac{2}{3} + \frac{4}{135n} + O\left(\frac{1}{n^2}\right), \quad n \rightarrow \infty.$$

This approximation of $S_n(1)$ is associated with a very elegant result communicated by Ramanujan to Hardy and later discussed at length by Watson [10].

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FIGURE 1. THE z -PLANE.

Buckholtz remarks that in view of (1.2) and (1.3), Ramanujan's approximation is a considerably more singular result than it would otherwise appear. Apart from the fact that the expansion (1.2) is not valid at $z = 1$, we note that it is also discontinuous across the curve C , the boundary of K and that the curve C passes through the point $z = 1$.

In this note we obtain an asymptotic expansion of $S_n(z)$ that holds uniformly in a neighborhood of $z = 1$. In particular, we investigate how the behavior of $S_n(z)$ changes across the curve C in that neighborhood. By the foregoing discussion we should expect $z = 1$ to be a singular point of the expansion. Consequently, the expansion is a generalized asymptotic expansion rather than an expansion of the Poincaré type. As in Watson [10] and Wong [11], we start with an integral representation of $S_n(z)$. By using the integral form of the remainder,

$$(1.5) \quad \begin{aligned} (nz)^n S_n(z) &= \int_0^{nz} (nz - t)^n e^t dt \\ &= (nz)^{n+1} \int_0^1 (1 - u)^n e^{nzu} du. \end{aligned}$$

By another change of variable,

$$(1.6) \quad \frac{S_n(z)}{nz} = \int_0^\infty e^{-n(v - e^\alpha + e^{\alpha-v})} e^{-v} dv, \quad z = e^\alpha.$$

The change from the z - to the α -plane makes the discussion much simpler later on. The point $z = 1$ corresponds to $\alpha = 0$. Therefore, we want the asymptotic expansion of the integral (1.6) in the neighborhood of $\alpha = 0$. Let

$$(1.7) \quad \psi(v) = v - e^\alpha + e^{\alpha-v}.$$

Since $\psi'(v) = 1 - e^{\alpha-v}$, $\psi''(v) = e^{\alpha-v}$, it follows that $\psi(v)$ has a simple saddle point at $v = \alpha$. As $\alpha \rightarrow 0$, the two critical points $v = \alpha$ and $v = 0$, that is, the saddle point and an end of the interval of integration, coalesce. A uniform asymptotic expansion of such an integral is obtained by first reducing

it to the standard form given by Bleistein [2]. However, after the standard form has been achieved, different techniques can be used to obtain the asymptotic expansion (see, for example, [9–11]). We use a technique developed recently in [7, 8]. This not only provides the expansion coefficients explicitly, but also simplifies the discussion of the nature of expansion.

We would like to emphasize two points regarding the asymptotic expansion given here. The first is that the singularity of the expansion is restricted to a single multiplicative factor; considering the fact that Bleistein's technique is applicable, this is not really surprising. The second is that the expansion holds for all z except the negative real axis. In view of the expansion given by Buckholtz, this is rather surprising. In §2, after some preliminary discussion, we state the main result. The proof is given in §§3 and 4.

2. MAIN RESULT

Before we can state our main result precisely, we need to reduce (1.6) to the standard form. Let

$$(2.1) \quad v - e^\alpha + e^{\alpha-v} = w^2/2 - bw,$$

$$(2.2) \quad b = \alpha \{2(e^\alpha - 1 - \alpha)/\alpha^2\}^{1/2}.$$

In (2.2) the principal branch of the square root function is taken. Furthermore, b is chosen so that the saddle point $v = \alpha$ in the v -plane corresponds to the saddle point $w = b$ in the w -plane. Now the mapping (2.1) can be given as

$$(2.3) \quad w - b = (v - \alpha) \left\{ \frac{2[e^{-(v-\alpha)} + (v - \alpha) - 1]}{(v - \alpha)^2} \right\}^{1/2}.$$

Again, we take the principal branch of the function. Thus $w - b \sim v - \alpha$ when $v \rightarrow \alpha$. By (2.1),

$$(2.4) \quad \frac{dw}{dv} = \frac{1 - e^{\alpha-v}}{w - b}.$$

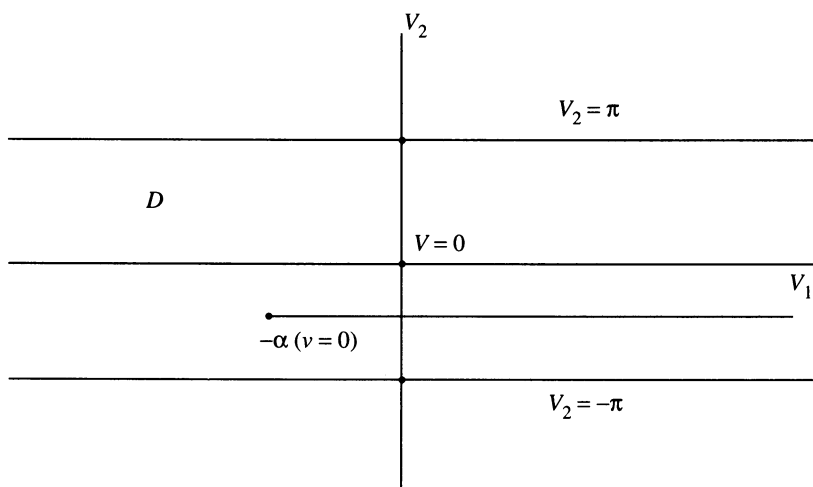
The point $v = \alpha$ is a regular point of the transformation given by (2.3). In fact, it can be shown that the mapping $w = w(v - \alpha)$ is one to one and maps the strip $|I_m(v - \alpha)| \leq \pi$ conformally on to a certain region in the w -plane. Let

$$(2.5) \quad V = V_1 + iV_2 = v - \alpha, \quad W = X + iY = w - b,$$

and let D denote the domain $|I_m V| = |I_m(v - \alpha)| \leq \pi$. The image of D in the W and the w -plane is denoted by D' and D'' respectively, (see Figures 2, 3, and 4). The boundary of the domain D , $V_2 = \pi$ and $V_2 = -\pi$ are mapped onto the hyperbolic curves $XY = \pi$, $Y > 0$ and $XY = -\pi$, $Y < 0$ respectively in the W -plane. The path of integration in each plane is indicated by an arrow.

By (1.6) and the transformation (2.1),

$$(2.6) \quad \begin{aligned} \frac{S_n(z)}{nz} &= \int_0^{\infty+b} e^{-n(w^2/2-bw)} e^{-v} \frac{dv}{dw} dw \\ &= \int_0^{\infty+b} e^{-n(w^2/2-bw)} g(w) dw, \end{aligned}$$

FIGURE 2. THE V -PLANE. $V = V_1 + iV_2 = v - \alpha$.

where

$$(2.7) \quad g(w) = (w - b)/(e^v - e^\alpha).$$

Since by (2.3),

$$g(b) = \lim_{w \rightarrow b} g(w) = e^{-\alpha} = \frac{1}{z},$$

we can write (2.6) as follows.

$$(2.8) \quad (nz)^{-1} S_n(z) - \frac{1}{z} H_n(b) = \int_0^{\infty+b} e^{-n(w^2/2-bw)} G(w)(w-b) dw,$$

where

$$(2.9) \quad H_n(b) = \int_0^\infty e^{-n(w^2/2-bw)} dw$$

and

$$(2.10) \quad G(w) = \{g(w) - g(b)\}/(w-b).$$

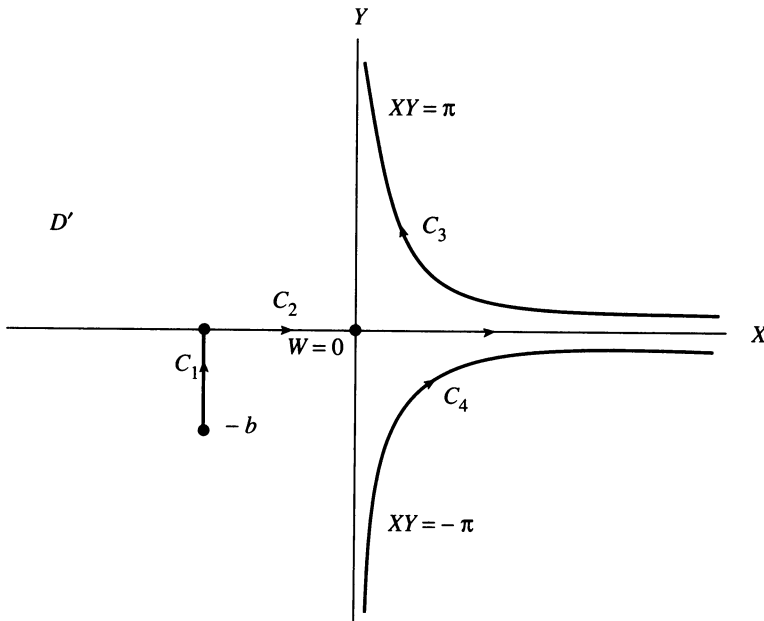
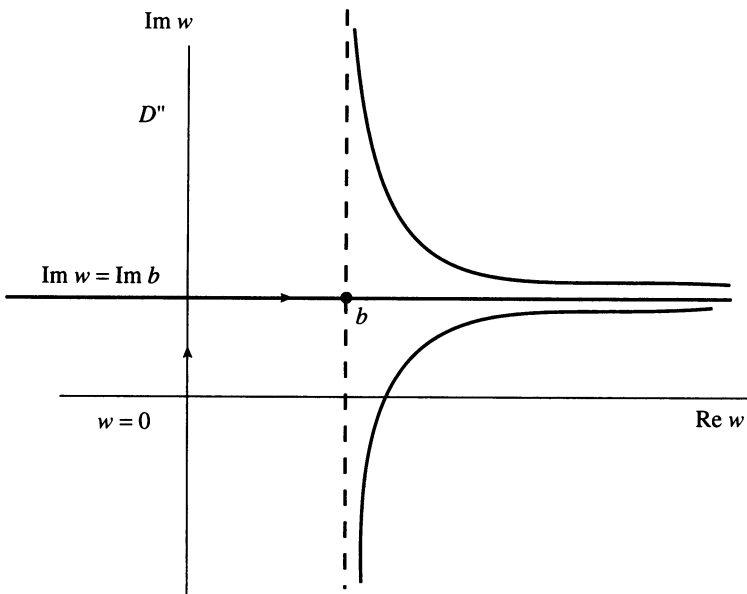
$H_n(b)$ is related to the complementary incomplete error function. The standard form of the integral mentioned earlier is given in (2.6), but (2.8) is more convenient for deriving the asymptotic expansion. $G(w)$ is analytic in the region, which in the $W = w - b$ -plane, is shown as region D' (see Figure 3). Now we expand $G(w)$ in terms of the polynomials $P_k(w)$, which are defined as follows:

$$(2.11) \quad \begin{aligned} P_0(w) &= 1, & P_1(w) &= w, \\ P'_k(w) &= (w-b)P_{k-2}(w), & P_k(0) &= 0, & k &= 2, 3, \dots \end{aligned}$$

These polynomials are discussed in a more general form in [7, 8]. Since $G(w)$ is analytic in the region that includes the origin, we can write

$$(2.12) \quad G(w) = \sum_{k=0}^{\infty} c_k P_k(w), \quad c_k = c_k(b).$$

This series converges in the neighborhood of the origin, at least for $|b|$ suffi-

FIGURE 3. THE W -PLANE.FIGURE 4. THE w -PLANE. $W = X + iY = w - b$, $-b = f(-\alpha)$.

ciently small. By using (2.11) and integration by parts successively, we see that for $k = 0, 1, 2, \dots$,

$$(2.13) \quad \int_0^\infty e^{-n(w^2/2-bw)} P_{2k}(w)(w-b) dw = n^{-k-1},$$

$$(2.14) \quad \int_0^\infty e^{-n(w^2/2-bw)} P_{2k+1}(w)(w-b)dw = n^{-k-1}H_n(b),$$

where $H_n(b)$ is defined by (2.9). Thus, by (2.8) and (2.12), we have the formal asymptotic expansion,

$$(2.15) \quad (nz)^{-1}S_n(z) \sim \left(\frac{1}{z} + \sum_{k=0}^\infty \frac{c_{2k+1}}{n^{k+1}}\right) H_n(b) + \sum_{k=0}^\infty \frac{c_{2k}}{n^{k+1}}.$$

Now we state the main result.

Theorem. *Let Ω be a compact subset of the complex plane with the cut, $|\arg z| < \pi$, $|z| > 0$. Then for all $z \in \Omega$,*

$$(2.16) \quad (nz)^{-1}S_n(z) = \left(\frac{1}{z} + \sum_{k=0}^{N-1} \frac{c_{2k+1}}{n^{k+1}}\right) H_n(b) + \sum_{k=0}^{N-1} \frac{c_{2k}}{n^{k+1}} + E_{2N}(z, n),$$

$$(2.17) \quad E_{2N}(z, n) = \int_0^{\infty+b} \left[G(w) - \sum_{k=0}^{2N-1} c_k P_k(w) \right] (w-b) e^{-n(w^2/2-bw)} dw,$$

$b = b(z)$ is determined uniquely by z and satisfies

$$(2.18) \quad e^{-b^2/2} = ze^{1-z}.$$

Furthermore, (i) the coefficients c_k are analytic functions of z in Ω , and (ii)

$$(2.19) \quad E_{2N}(z, n) = O(n^{-N} \max\{1, |H_n(b)|\}), \quad n \rightarrow \infty,$$

uniformly in Ω .

We will prove this in the next two sections. Here we would like to make a remark about the expansion. Since $c_k(b)$ are analytic functions of z , by (2.15) it is clear that the only singularity in the asymptotic expansion is due to the multiplicative factor $H_n(b)$. By [5, pp. 311–312; 1, p. 300], $H_n(b) = O(n^{-1/2})$ for $b = O(n^{-1/2})$ and for $|bn^{1/2}| \gg 1$,

$$(2.20) \quad H_n(b) \sim \begin{cases} e^{nb^2/2} \left(\frac{2\pi}{n}\right)^{1/2} & |\arg b| < \frac{\pi}{4}, \\ (-nb)^{-1} & \frac{\pi}{4} \leq |\arg b| \leq \pi. \end{cases}$$

If $b = \sigma + i\tau$, the halflines $\sigma = \tau$ and $\sigma = -\tau$, $\sigma > 0$ form the boundary of the region in which the part of the asymptotic expansion with $H_n(b)$ as a factor dominates. Note that by (2.18), $|e^{-b^2/2}| = e^{-(\sigma^2 - \tau^2)/2} = |ze^{1-z}|$. Therefore the boundary D of the region K in the expansion given by Buckholtz, corresponds to the halflines $\sigma = |\tau| > 0$.

3. THE EXPANSION COEFFICIENTS

Let $V = v - \alpha$, $W = w - b$ and

$$(3.1) \quad f(\xi) = \xi \{2(e^{-\xi} + \xi - 1)/\xi^2\}^{1/2}, \quad |I_m(\xi)| \leq \pi.$$

By (2.3), $W = f(V)$, $V \in D$ and $D' = f(D)$. As remarked earlier, f is analytic and one to one. Also, by (2.2), $b = -f(-\alpha)$, $\alpha = \text{Log } z$. Therefore,

b is an analytic function of z in $|\arg z| < \pi$ and mapping is one to one. If z belongs to Ω , then $-b$ is in the interior of D' and b is in the interior of D'' (see Figure 4). To prove that c_k are analytic functions of z , we only need to show that they are analytic functions of b . The c_k are computed as follows (see [8, Theorem 5], $\gamma = 0$)

$$(3.2) \quad c_k(b) = \frac{1}{2\pi i} \int_{C_w} G(\zeta) \phi_k(\zeta) d\zeta,$$

where C_w is a simple, closed contour in the w -plane in the region D'' enclosing the points $w = 0$ and $w = b$. The functions $\phi_n(\zeta)$ are generated by the expansion

$$(3.3) \quad \frac{1}{\zeta - w} = \sum_{k=0}^{\infty} P_k(w) \phi_k(\zeta).$$

They satisfy the following relation

$$\phi_0(\zeta) = \frac{1}{\zeta}, \quad \phi_1(\zeta) = \frac{1}{(\zeta - b)^2}, \quad \phi_{k+2}(\zeta) = -\frac{d}{d\zeta} \frac{\phi_k(\zeta)}{(\zeta - b)}.$$

The coefficients c_k are computed by using the representations [8, (3.1) and (3.2)],

$$(3.4) \quad \phi_{2k-1}(\zeta) = \frac{(2k)!}{k! 2^k (\zeta - b)^{2k}},$$

$$(3.5) \quad \phi_{2k}(\zeta) = \sum_{m=0}^k \frac{(k+m)!}{m! 2^m \zeta^{k-m+1} (\zeta - b)^{k+m}}.$$

We can choose the contour C_w in D'' such that $\{b : b = -f(-\alpha), z = e^\alpha, z \in \Omega\}$ and $w = 0$ are inside it. By (2.7) and (2.10), $w = b$ is a regular point of G . Since

$$(3.6) \quad G(w) = (e^v - e^\alpha)^{-1} - \frac{1}{z}(w - b)^{-1},$$

the only singularities of G are at the points that correspond to $v = \alpha + 2k\pi i$, $k = \pm 1, \pm 2, \dots$. All such points are outside D'' . Hence the analyticity of $c_k(b)$ follows from the representation (3.2). These coefficients can be obtained by the residue theorem. Near $v = \alpha$,

$$(3.7) \quad \begin{aligned} v - \alpha &= (w - b) + \frac{1}{6}(w - b)^2 + \frac{1}{36}(w - b)^3 \\ &+ \frac{1}{270}(w - b)^4 + \frac{1}{4320}(w - b)^5 + \dots \end{aligned}$$

and near $v = 0$,

$$(3.8) \quad v = b(e^\alpha - 1)^{-1}w + 2^{-1}(e^\alpha - 1)^{-3}\{e^\alpha b^2 - (e^\alpha - 1)^2\}w^2 + \dots.$$

The first four coefficients in the asymptotic expansion are,

$$(3.9) \quad \begin{aligned} c_0(b) &= -(e^\alpha - 1)^{-1} + \frac{e^{-\alpha}}{b}, \quad c_1(b) = \frac{e^{-\alpha}}{12}, \\ c_2(b) &= (e^\alpha - 1)^{-3} + e^{-\alpha} \left\{ \frac{1}{12b} - \frac{1}{b^3} \right\}, \quad c_3(b) = \frac{e^{-\alpha}}{288}. \end{aligned}$$

By using the expansion for $H_n(b)$ when $|\arg b| > \frac{\pi}{4}$, we obtain

$$S_n(z) = \frac{z}{1-z} + \frac{1}{n} \frac{z}{(1-z)^3} + \cdots$$

These are the first two terms of Buckholtz's expansion. As $z \rightarrow 1$, $\alpha \rightarrow 0$ and therefore, $b \rightarrow 0$. By taking the limit as $\alpha \rightarrow 0$, $c_0(0) = -\frac{2}{3}$, $c_1(0) = \frac{1}{12}$, $c_2(0) = \frac{4}{135}$, and $c_3(0) = \frac{1}{288}$. Also $H_n(0) = (\frac{\pi}{2n})^{1/2}$. The expansion at $z = 1$ reduces to

$$S_n(1) = \left(\frac{\pi n}{2}\right)^{1/2} \left\{ 1 + \frac{1}{12n} + \frac{1}{288n^2} \right\} - \frac{2}{3} + \frac{4}{135n} + \cdots,$$

which agrees with the form of $S_n(1)$ given in [3].

4. THE REMAINDER E_{2N}

We will prove that the bound for $E_{2N}(z, n)$ given in (2.19) holds uniformly in z , $z \in \Omega$. Without loss of generality we may assume that $\Omega = \{z/|\arg z| \leq \pi - \delta \text{ and } \Delta^{-1} \leq |z| \leq \Delta\}$. Then $|\operatorname{Re} \alpha| \leq \ln \Delta$ and $|\operatorname{Im} \alpha| \leq \pi - \delta$. Denote this set by Ω' and let $\Omega'' = f(\Omega')$ so that $-b$ belongs to Ω'' . Since f maps the real V -axis onto the real W -axis, Ω'' is symmetric about the real W -axis. Let

$$(4.1) \quad R_{2N}(w) = G(w) - \sum_{k=0}^{2N-1} c_k P_k(w).$$

Now we use some recent results concerning the polynomials P_k and the associated functions ϕ_k . By [7, (7.5) and (7.6) with $\gamma = 0$], (2.17) can be written as follows:

$$(4.2) \quad \begin{aligned} E_{2N}(z, n) &= n^{-N} \int_0^{\infty+b} \mathcal{D}^N[R_{2N}(w)](w-b)e^{-n(w^2/2-bw)} dw \\ &= n^{-N} e^{nb^2/2} \int_{-b}^{\infty} \mathcal{D}^N[R_{2N}(W+b)]W e^{-nW^2/2} dW, \end{aligned}$$

$$\mathcal{D}R_{2N}(w) = (w-b)^{-1} \frac{d}{dw} R_{2N}(w), \quad \mathcal{D}^m = \mathcal{D}\mathcal{D}^{m-1}, \quad m = 2, 3, \dots$$

We can replace the line of integration in the last integral by the contours C_1 and C_2 where C_1 is the straight line segment joining $-b = -\sigma - i\tau$ and $-\sigma + i0$ and C_2 is the halfline $x = \operatorname{Re} W \geq -\sigma$. Let us assume for the present that $\mathcal{D}^N R_{2N}$ is bounded on C_1 and C_2 . Then

$$(4.3) \quad \begin{aligned} \int_{C_1} \mathcal{D}^N R_{2N}(W+b)W e^{-nW^2/2} dW &= O\left(\int_0^\tau (\sigma^2 + t^2)^{1/2} e^{-n(\sigma^2 - t^2)/2} dt\right), \\ &= O(e^{-n(\sigma^2 - \tau^2)/2}), \quad n \rightarrow \infty. \end{aligned}$$

Since $-b$ belongs to the compact set Ω'' , the implied constant in (4.3) can be

taken independently of σ and τ . Again,

$$(4.4) \quad \int_{C_2} \mathcal{D}^N R_{2N}(W+b) W e^{-nW^2/2} dW = O\left(\int_{-\sigma}^{\infty} |X| e^{-nX^2/2} dX\right) \\ = \begin{cases} O\left(\frac{1}{n}\right) & \sigma \geq 0, \\ O\left(\frac{1}{n} e^{-n\sigma^2/2}\right) & \sigma < 0. \end{cases}$$

In each case, the implied constant is independent of b . By (4.3) and (4.4) we obtain (2.19). To complete the proof we need to show that $\mathcal{D}^N R_{2N}(W+b)$ is indeed bounded for all W on C_1 and C_2 . Again, we use the technique developed in [8]. By [8, (6.17)],

$$(4.5) \quad \mathcal{D}^N[R_{2N}(w)] = \frac{1}{2\pi i} \int_{C_w} \Theta_{\zeta}^N(\zeta-w)^{-1} G(\zeta) d\zeta,$$

where C_w is a contour in the w -plane within D'' and enclosing the points $\zeta = 0$, $\zeta = b$, and $\zeta = w$; $\Theta_{\zeta}(\zeta-w)^{-1} = -\frac{d}{d\zeta}(\zeta-b)^{-1}(\zeta-w)^{-1}$, $\Theta_{\zeta}^m = \Theta_{\zeta} \Theta_{\zeta}^{m-1}$. By [8, Lemma 3],

$$(4.6) \quad \Theta_{\zeta}^N(\zeta-w)^{-1} = \sum_{m=0}^N \frac{(N+m)!}{m! 2^m (\zeta-b)^{N+m} (\zeta-w)^{N-m+1}}.$$

Therefore, by a change of variable,

$$(4.7) \quad \mathcal{D}^N[R_{2N}(W+b)] = \sum_{m=0}^N \frac{(N+m)!}{m! 2^m} \int_{C_w} \frac{G(\eta+b)}{\eta^{N+m}(\eta-W)^{N-m+1}} d\eta;$$

where $\eta = \zeta - b$, $W = w - b$, and C_w is the contour in the W -plane corresponding to C_w in the w -plane. Note that by (2.7) and (2.10),

$$(4.8) \quad G(W+b) = \frac{1}{z} \left\{ \frac{1}{e^V - 1} - \frac{1}{W} \right\}.$$

$G(W+b)$ is not only analytic in D' , it is also bounded. On $V = V_1 \pm i\pi$, $-\infty < V_1 < \infty$, $(e^V - 1)^{-1}$ is bounded. By the Phragmén-Lindelöf theorem and the maximum principle, $G(W(V)+b) = G(W+b)$ is bounded for $V \in D$ or $W \in D'$. Now consider the boundaries of D and D' . The boundary of D' is the map of $V = V_1 \pm i\pi$. Let $W = X + iY$. By (2.3) and (3.1), $V = V_1 \pm i\pi$,

$$(4.9) \quad X^2 - Y^2 = 2(-e^{-V_1} + V_1 - 1), \quad XY = \pm\pi.$$

The boundary curves of D' are, $XY = \pi$, $Y > 0$ and $XY = -\pi$, $Y < 0$. By considering the mapping (2.3), it follows that $V_1 \rightarrow \infty$ if and only if $X \rightarrow \infty$. By (4.9), $V_1 \sim X^2/2$, $X \rightarrow \infty$. Therefore, on the curves $XY = \pm\pi$,

$$(4.10) \quad |(e^V - 1)^{-1}| = (e^{V_1} + 1)^{-1} \leq e^{-X^2/4}, \quad X \gg 1.$$

To show that $D^N R_{2N}(W+b)$ is bounded for all W on C_1 and C_2 , consider the integral

$$(4.11) \quad zI = \int_{C_w} \frac{zG(\eta+b)}{\eta^{N+m}(\eta-W)^{N-m+1}} d\eta, \\ = \int_{C_w} \frac{1}{\eta^{N+m}(\eta-W)^{N-m+1}(e^{V(\eta)} - 1)} d\eta, \quad m = 0, 1, \dots, N.$$

For any fixed point W on C_1 or C_2 , the contour C_W has the points $\eta = 0, b, W$ in its interior. We can replace C_W by the contours C_3 , $XY = \pi$, $Y > 0$ and C_4 , $XY = -\pi$, $Y < 0$. Let J denote the above integral along the curve C_3 , $\eta = X + iY$, $XY = \pi$, $Y > 0$, $dY = (-\pi/X^2) dX$. Then,

$$(4.12) \quad |J| \leq M \int_0^\infty \left(X^2 + \frac{\pi^2}{X^2} \right)^{-(N+m)/2} |X + iY - W|^{-N+m-1} e^{-X^2/4} \left(\frac{X^2 + \pi}{X^2} \right) dX.$$

Note that $(X^2 + \pi^2/X^2) \geq 2\pi$. Let d denote the distance function. If W is on C_1 , then by symmetry, $d(W, C_3) \geq d(-b, C_3 \cup C_4)$. Since $-b \in \Omega''$ and $d(\Omega'', C_3 \cup C_4) = \gamma > 0$, it follows that $d(W, C_3) \geq \gamma$. If W is on C_2 , let $W = t$, $-\sigma \leq t < \infty$, then $|X + iY - W|^2 \geq \pi^2/X^2$. Therefore, for $m = 0, 1, \dots, N \geq 2$,

$$(4.13) \quad |J| \leq M'(m) \int_0^\infty \frac{X^2 + \pi}{X^4 + \pi^2} \left\{ \gamma^{-N+m-1} + \left(\frac{X}{\pi} \right)^{N+1-m} \right\} e^{-X^2/4} dX \\ \leq M''(m, \gamma).$$

A similar bound holds for the integral along C_4 . Since $\frac{1}{z}$ is bounded for $z \in \Omega$, by (4.11), $|I| \leq M'''(m, \gamma)$ for some constant M''' . This completes the proof.

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