COMPOSITIONS OF CONTINUOUS FUNCTIONS AND CONNECTED FUNCTIONS

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ABSTRACT. Suppose $f: X \rightarrow Y$ is continuous and onto and $g: Y \rightarrow Z$ is such that $g \circ f: X \rightarrow Z$ has a property we are interested in. For which properties of functions can we infer that g has the same property? Properties for which we can infer this include continuity and the Darboux property. Properties for which we cannot include almost continuity.

In calculus, we encounter the result that the composition $g \circ f$ of two continuous functions $f: I \rightarrow I$ and $g: I \rightarrow I$ is continuous, where $I = [0, 1]$. This brings to mind the following question: If $f: X \rightarrow Y$ is continuous and onto, what property should $g \circ f$ have in order for $g: Y \rightarrow Z$ to possess the same property? For example, if f and $g \circ f$ are continuous, will g be continuous? The reason we suppose f is onto Y is that $g \circ f$ would be continuous when g is continuous on $f(X)$, but not on $X \setminus f(X)$. Of course, there are many properties of functions for which this question is of interest. We consider only a few of our favorites here.

If h is any function, we let $\text{Gr}(h)$ denote the graph of h . Suppose $f: X \rightarrow Y$ is continuous and onto and $g: Y \rightarrow Z$. The height-preserving function $H: X \times Z \rightarrow Y \times Z$ defined by $H(x, y) = (f(x), y)$ is continuous. Now,

$$\begin{aligned} H(\text{Gr}(g \circ f)) &= H(\{(x, g(f(x))) : x \in X\}) \\ &= \{(f(x), g(f(x))) : x \in X\} = \text{Gr}(g) \end{aligned}$$

because f is onto X . This observation immediately gives us the following two theorems, the first being an easy consequence of the closed graph theorem.

Theorem 1. *If $f: X \rightarrow Y$ is continuous and onto and if $g \circ f: X \rightarrow Z$ is continuous where X , Y , and Z are compact, then $g: Y \rightarrow Z$ is continuous.*

Theorem 2. *If $f: X \rightarrow Y$ is continuous and onto and if $g \circ f: X \rightarrow Z$ has a connected graph, then $g: Y \rightarrow Z$ has a connected graph.*

However, if $f: X \rightarrow Y$ is not an identification, then according to Theorem 3.1 of [1], there exist a space Z and a function $g: Y \rightarrow Z$ where g is not continuous but $g \circ f$ is continuous.

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Let $\mathfrak{R}$ denote the set of real numbers with the usual topology.

A function is said to be a *Darboux* function if the image of each connected set is connected.

Theorem 3. Suppose X is a metric continuum and $f: X \rightarrow I$ is continuous and onto. If $g \circ f: X \rightarrow \mathfrak{R}$ is a Darboux function, then so is $g: I \rightarrow \mathfrak{R}$.

Proof. Suppose $a < b$ and z lies between $g(a)$ and $g(b)$. We show there exists a number c between a and b such that $g(c) = z$. Let N be a component of $f^{-1}(a, b)$. According to Theorem 2-16 of [7], $\text{cl}(f^{-1}(a, b)) \setminus f^{-1}(a, b)$ contains a limit point of N . Therefore, since f is continuous, $\text{cl}(N)$ meets either $f^{-1}(a)$ or $f^{-1}(b)$. Let $\mathcal{E}$ denote the collection of all components C of $f^{-1}(a, b)$ such that $\text{cl}(C)$ meets $f^{-1}(a)$, and let $\mathcal{D}$ denote the collection of all components D of $f^{-1}(a, b)$ such that $\text{cl}(D)$ meets $f^{-1}(b)$.

Here is how to obtain a connected set M in $f^{-1}[a, b]$ that meets both $f^{-1}(a)$ and $f^{-1}(b)$. In case there is a member C of $\mathcal{E} \cap \mathcal{D}$, let $M = \text{cl}(C)$. Suppose $\mathcal{E} \cap \mathcal{D} = \emptyset$. $H = f^{-1}[0, a]$ and $K = f^{-1}[b, 1]$ are separated sets. Let $A = H \cup (\bigcup \mathcal{E})$ and $B = K \cup (\bigcup \mathcal{D})$. Since X is connected and $X = A \cup B$, we may suppose, for argument's sake, that $B \cap \text{cl}(A) \neq \emptyset$. Then there is a point $p \in B \cap \text{cl}(A)$. Let $\{x_n\}$ be a sequence in $\bigcup \mathcal{E}$ converging to p , and let C_n be the member of $\mathcal{E}$ to which x_n belongs. For each n , choose $y_n \in f^{-1}(a) \cap \text{cl}(C_n)$. For simplicity, we may suppose $\{y_n\}$ already converges to a point q . Then $q \in f^{-1}(a)$. By Theorem 2-101 of [7], $Y = \limsup C_n$ is connected, and $p, q \in Y$. In case $p \in K$, then Y meets $f^{-1}(a)$ and $f^{-1}(b)$, so choose $M = Y$. In case $p \in D_n$ for some $D_n \in \mathcal{D}$, choose M to be the connected subset $Y \cup \text{cl}(D_n)$ of $f^{-1}[a, b]$ which then meets both $f^{-1}(a)$ and $f^{-1}(b)$.

Now, $f(M) = [a, b]$. Since $g \circ f$ is a Darboux function, $(g \circ f)(M)$ is connected, and $(g \circ f)(M) = g[a, b]$. Therefore there exists a number $w \in M$ such that $g(f(w)) = z$. Choosing $c = f(w)$, we have $a < c < b$ and $g(c) = z$.

A function $f: X \rightarrow Y$ is *almost continuous* if, whenever $D \subseteq X \times Y$ is an open set with $\text{Gr}(f) \subseteq D$, then there exists a continuous function $g: X \rightarrow Y$ such that $\text{Gr}(g) \subseteq D$.

Example 1. There exist $f: I \rightarrow I$ continuous and onto and $g: I \rightarrow I$ not almost continuous such that $g \circ f$ is almost continuous.

Proof. In [8] F. B. Jones and E. S. Thomas, Jr. give an example of a function from I to I with a connected graph which is not almost continuous. This function will be the function g in our example. Let C denote the Cantor middle-third set. A property of g that we will use is that if (a, b) is a component of $I \setminus C$, then $g|_{[a, b]}$ is continuous and $g(a, b) = [0, 1]$.

The continuous function f is defined to be the identity on C . Also, if (a, b) is a component of $I \setminus C$, then f maps (a, b) onto itself and has an N shape. More precisely, for each such (a, b) , choose c and d such that $a < c < d < b$. Define f to be continuous on $[a, b]$ so that $f(a) = f(d) = a$, $f(c) = f(b) = b$, f is increasing on $[a, c]$ and $[d, b]$ and is decreasing on $[c, d]$.

Observe that if (a, b) is a component of $I \setminus C$, then $g \circ f|_{[a, b]}$ is continuous and there exist disjoint subintervals $[p, q]$ and $[r, s]$ of $[a, b]$ such that $g(f(p)) = g(f(s)) = 0$, and $g(f(q)) = g(f(r)) = 1$.

We claim that $g \circ f$ is almost continuous. Assume that it is not. From the definition of almost continuity, there exists a closed set K in I^2 such that $K \cap \text{Gr}(g \circ f) = \emptyset$ and if $h: I \rightarrow I$ is continuous, then $\text{Gr}(h) \cap K \neq \emptyset$. We may take K to be irreducible with respect to the property of intersecting all graphs of continuous functions from I into I (see [9]), from which it follows that the X -projection, $\pi_X(K)$, of K is connected. Since $g \circ f$ is continuous on $I \setminus C$, it is clear that $\pi_X(K)$ cannot be contained in the closure of any component of $I \setminus C$. Thus, there is a point z of C in the interior of $\pi_X(K)$, where z is not an endpoint of a component of $I \setminus C$. By the irreducibility of K , there exist continuous functions $i: [0, z] \rightarrow I$ and $j: [z, 1] \rightarrow I$ such that neither $\text{Gr}(i)$ nor $\text{Gr}(j)$ contains a point of K . Either $i(z) > j(z)$ or $i(z) < j(z)$. As the two cases are similar, we assume that $i(z) > j(z)$. Choose a circular open neighborhood U of $(z, j(z))$ such that $U \cap K = \emptyset$. Note that $\text{Gr}(i) \cap U = \emptyset$, for otherwise we could easily construct a continuous function on I whose graph misses K . We may choose a component (a, b) of $I \setminus C$ where $(a, b) \subseteq \pi_X(U)$ and $b < z$. There exists an interval $[r, s]$ in (a, b) such that $g(f(r)) = 1$ and $g(f(s)) = 0$. We now construct the graph of a continuous function on I by taking the union of parts of the graphs of four functions. Take the part of $\text{Gr}(i)$ from $(0, i(0))$ to a point $(c, i(c))$ where $i(c) = g(f(c))$ and $c \in [r, s]$. Take the part of $\text{Gr}(g \circ f)$ from $(c, g(f(c)))$ down to a point $(d, g(f(d))) \in U$, with $d < z$. Now take a straight line segment from $(d, g(f(d)))$ to $(z, j(z))$, together with all of $\text{Gr}(j)$. Since the graph of the resulting continuous function does not meet K , we have a contradiction. Thus $g \circ f$ is almost continuous.

Let $f: I \rightarrow I$ be continuous, $g: I \rightarrow I$ have a connected graph, and $h: I \rightarrow I$ be almost continuous. It is known that both $h \circ f$ and $f \circ h$ are almost continuous [10] and that $f \circ g$ has a connected graph [6]. We show next that $g \circ f$ has a connected graph, too. This result seems to have been overlooked. In Theorem 2.7 of [6], Hildebrand and Sanderson did show that given topological spaces R and T , a set S , and functions $f: R \rightarrow S$ and $g: S \rightarrow T$, then $g \circ f$ is a connectivity function if and only if S has a topology for which f is a connectivity function and g is continuous.

Theorem 4. *If $f: \mathfrak{R} \rightarrow \mathfrak{R}$ is continuous and $g: \mathfrak{R} \rightarrow \mathfrak{R}$ has a connected graph, then $g \circ f$ has a connected graph.*

Proof. According to [2], a function $g: \mathfrak{R} \rightarrow \mathfrak{R}$ with a connected graph can be characterized in terms of compact, connected subsets of the x, y plane $\mathfrak{R}^2$ as follows: The graph of g is connected if and only if whenever D is a continuum in $\mathfrak{R}^2$ which contains points above and below the graph of g , then D meets the graph of g . With this in mind, we let M be a compact, connected subset of $\mathfrak{R}^2$ which contains points above and below $\text{Gr}(g \circ f)$. That is, there exist points $(a, z_1), (b, z_2) \in M$ such that $z_1 > g(f(a))$ and $z_2 < g(f(b))$. (It may happen that $a = b$ if, for instance, M is a circle whose interior contains the point $(a, g(a))$.) Since the function $H: \mathfrak{R}^2 \rightarrow \mathfrak{R}^2$ defined by $H(x, y) = (f(x), y)$ is continuous, $H(M)$ is compact and connected. Let D denote $H(M)$. The points $H(a, z_1)$ and $H(b, z_2)$ of D lie, respectively, above the graph of g and below the graph of g . As a result, D intersects the graph of g . Let $(c, g(c)) \in D$. So $(c, g(c)) = (f(x_0), y_0)$ for some point $(x_0, y_0) \in M$. But $y_0 = g(c) = g(f(x_0))$, which implies $(x_0, y_0) \in \text{Gr}(g \circ f)$,

too. This shows that M meets the graph of $g \circ f$, and, consequently, $g \circ f$ has a connected graph.

$f: X \rightarrow Y$ is a *connectivity function* if $\text{Gr}(f|_C)$ is connected whenever C is connected. A function with domain $\mathfrak{R}$ is a connectivity function if and only if its graph is connected. However, Hagan [5] and Whyburn [11] have shown that a function $f: I^n \rightarrow I^m$, where $n \geq 2$, is a connectivity function if and only if f is peripherally continuous. That f is *peripherally continuous* at x means that given open neighborhoods U and V of x and $f(x)$, respectively, there exists a neighborhood W of x such that $x \in W \subseteq U$ and $f(\text{boundary}(W)) \subseteq V$.

It is somewhat surprising that Theorem 4 holds. Indeed Stallings [10] stated that connectivity functions do not, in general, satisfy a proposition similar to our Theorem 4 and that for this reason, the study of connectivity maps is difficult. In [10, §6, p. 261] Stallings indicated how to construct a counterexample. We give two examples below for completeness.

Example 2. Define $f: I^2 \rightarrow I$ by $f(x, y) = x$ and $g: I \rightarrow I$ by

$$g(x) = \begin{cases} (1 + \sin(2\pi/x))/2 & \text{if } 0 < x \leq 1; \\ 1 & \text{if } x = 0. \end{cases}$$

Now, let

$$X = (I \times \{0\}) \cup \bigcup_{i=1}^{\infty} (\{1/i\} \times I) \cup \{(0, 1)\}$$

and let $\hat{f}$ be the restriction of f to X . Then $\hat{f}$ is continuous, $\text{Gr}(g)$ is connected, but the point $(0, 1, 1)$ is an isolated point of $\text{Gr}(g \circ \hat{f})$. It follows that $g \circ f$ is not a connectivity function.

Example 3. There exist a continuous function $f: I^2 \rightarrow I^2$ and a peripherally continuous function $g: I^2 \rightarrow I$ such that $g \circ f$ is not peripherally continuous.

Proof. Let A be an arc in I^2 and let $f: I^2 \rightarrow I^2$ shrink A to a point. That is, f is continuous, $f(A)$ is a point, and f is a homeomorphism off A . Define $g: I^2 \rightarrow I$ by

$$g(P) = \begin{cases} (1 + \sin(1/\text{dist}(P, f(A))))/2 & \text{if } P \neq f(A); \\ 1/2 & \text{if } P = f(A). \end{cases}$$

Then g is peripherally continuous but $g \circ f$ is not.

Gibson and Roush [3] have proved that if $f: I^2 \rightarrow I$ is an onto continuous function and $g \circ f$ is a connectivity function, where $g: I \rightarrow I$, then g is continuous except perhaps at 0 or 1 (also see [4]). It follows from our Theorem 2 that g also has a connected graph.

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