ALL THE INFINITE NEST ALGEBRAS ARE ISOMORPHIC

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ABSTRACT. All the infinite nest algebras in a separable Hilbert space are completely isomorphic.

In this note we prove that all the infinite nest algebras in a separable Hilbert space are completely isomorphic to each other. The simplest candidate resembles one of the isomorphic representations of H^∞ given by Wojtaszczyk [W]. He proved that H^∞ is isomorphic to $(\sum_{i=1}^n \oplus A_n)_\infty$ where $A_n = \text{span}\{z^i\}_{i=0}^n \subset A$ and A is the disc algebra.

For this paper H denotes a separable Hilbert space. A *nest of projections* in H is any linearly ordered set $\mathcal{P}$ of orthogonal projections closed in the strong operator topology and containing 0 and I . The *nest algebra* induced by $\mathcal{P}$ is the set of all operators T in $B(H)$ that leaves invariant every element of $\mathcal{P}$; i.e.,

$$\text{Alg } \mathcal{P} = \{T \in B(H) : (I - P)TP = 0 \text{ for every } P \in \mathcal{P}\}.$$

Two of the most common nests of projections are the following:

The discrete nest. Let (e_i) be an orthonormal basis for H , and for $0 \leq n \leq \infty$ let P_n be the orthogonal projections onto $\overline{\text{span}}\{e_i : 1 \leq i \leq n\}$.

The Volterra nest. Let $H = L_2[0, 1]$, and for $0 \leq t \leq 1$ let P_t be the orthogonal projection onto the subspace of all functions supported on $[0, t]$.

These examples are fundamental and have motivated a big part of the theory. If $\mathcal{P}$ is the discrete nest, $\text{Alg } \mathcal{P}$ is the set of all "upper triangular" operators with respect to the orthonormal basis; and the Volterra nest is the canonical example of a *continuous* nest (without gaps). Indeed, Larson [L] proved that any two continuous nests are similar; i.e., If $\mathcal{P}$ and $\widetilde{\mathcal{P}}$ are continuous nests, then there exists $V \in B(H)$ invertible such that $\text{Alg } \mathcal{P} = V^{-1} \text{Alg } \widetilde{\mathcal{P}} V$. Such an isomorphism is called *spatial*.

We distinguish between *algebraic isomorphism* and *Banach space isomorphism* between nest algebras. The first one preserves the linear and the algebraic properties of the nest algebra.

Ringrose [R] proved that every algebraic isomorphism between nest algebras is spatial; and it is well known that the discrete and the Volterra nest algebras

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are not similar. Nevertheless, they cannot be distinguished by their Banach space structure.

Let $V, W \subset B(H)$ and $u: V \rightarrow W$ a linear map. Define

$$u_n: M_n(V) \rightarrow M_n(W) \quad \text{by} \quad u_n([a_{ij}]) = [u(a_{ij})].$$

Since $M_n(V), M_n(W) \subset M_n(B(H)) \equiv B(H^n)$, they have a natural norm induced by $B(H^n)$. We say that u is completely bounded if $\|u\|_{cb} = \sup_n \|u_n\| < \infty$. Moreover, if both u and u^{-1} are completely bounded, then u is a complete isomorphism.

The main result is the following:

Theorem 1. *Let $\text{Alg } \mathcal{P}$ be an infinite nest algebra in a separable Hilbert space. Then $\text{Alg } \mathcal{P}$ is completely isomorphic to $(\sum_{n=1}^{\infty} \oplus \mathcal{T}_n)_{\infty}$, where $\mathcal{T}_n$ is the algebra of all upper triangular $n \times n$ matrices.*

The proof of Theorem 1 follows from Pelczyński's decomposition method (see [Pe1]) and the semidiscreteness of the nest algebras proved by Paulsen, Power, and Ward [PPW].

Theorem 2 (Paulsen, Power, and Ward [PPW]). *Let $\mathcal{A} = \text{Alg } \mathcal{P}$ be a nest algebra acting on a separable Hilbert space H . Then there exist*

- (i) *a sequence $\mathcal{A}_n$ of finite-dimensional nest algebras;*
- (ii) *σ -weakly continuous completely contractive maps $\phi_n: \mathcal{A} \rightarrow \mathcal{A}_n$;*
- (iii) *σ -weakly continuous completely isometric homomorphisms $\psi_n: \mathcal{A}_n \rightarrow \mathcal{A}$, such that $\psi_n(\phi_n(A)) \rightarrow A$ σ -weakly for all A in $\mathcal{A}$.*

Pelczyński's decomposition method says that if X and Y are Banach spaces and if X embeds complementably in Y , Y embeds complementably in X and X is isomorphic to its l_{∞} -sum; i.e., $X \approx (\sum \oplus X)_{l_{\infty}}$, then X is isomorphic to Y . Moreover, if the embeddings are completely bounded and completely complementable and the isomorphism is a complete isomorphism then X and Y are completely isomorphic.

It is clear that $(\sum_{n=1}^{\infty} \oplus \mathcal{T}_n)_{\infty}$ is completely isomorphic to its l_{∞} -sum. Moreover, if $\mathcal{P}$ is infinite it is easy to see that $(\sum_{n=1}^{\infty} \oplus \mathcal{T}_n)_{\infty}$ embeds completely bounded and completely complemented in $\text{Alg } \mathcal{P}$. To see this, notice that we need only n elements of $\mathcal{P}$ to embed $\mathcal{T}_n$ in $\text{Alg } \mathcal{P}$. It remains to prove that $\text{Alg } \mathcal{P}$ embeds completely bounded and completely complemented in $(\sum_{n=1}^{\infty} \oplus \mathcal{T}_n)_{\infty}$.

We prove Theorem 1 first for multiplicity free nests. The advantage here is that the finite nest algebras $\mathcal{A}_n$ of Theorem 2 are the $\mathcal{T}_n$'s. This is immediate from the proof of Theorem 2.

Lemma 3. *Let $\text{Alg } \mathcal{P}$ be an infinite multiplicity free nest algebra in a separable Hilbert space. Then $\text{Alg } \mathcal{P}$ is completely isomorphic to $(\sum_{n=1}^{\infty} \oplus \mathcal{T}_n)_{\infty}$, where $\mathcal{T}_n$ is the algebra of all upper triangular $n \times n$ matrices.*

Proof. Use Theorem 2 and the previous remark to find for every n , maps $\phi_n: \text{Alg } \mathcal{P} \rightarrow \mathcal{T}_n$ and $\psi_n: \mathcal{T}_n \rightarrow \text{Alg } \mathcal{P}$ with the required properties. Define $\Phi: \text{Alg } \mathcal{P} \rightarrow (\sum_{n=1}^{\infty} \oplus \mathcal{T}_n)_{\infty}$ by

$$\Phi(T) = (\phi_n(T))_n.$$

By Theorem 2 (ii), (iii), one has that Φ is completely isometric. It remains to prove that the embedding is completely complemented. Let $\mathcal{U}$ be a free ultrafilter on the natural numbers and define $\Psi: (\sum_{n=1}^{\infty} \oplus \mathcal{T}_n)_{\infty} \rightarrow \text{Alg } \mathcal{P}$ by

$$\Psi((T_n)) = \lim_{n \in \mathcal{U}} \psi_n(T_n),$$

where the limit is taken in the σ -weak topology.

It is easy to see that Ψ is well defined and is completely bounded. Theorem 2(iii) implies says that $\Psi\Phi = \text{id}$. Then $\Phi\Psi$ gives the desired projection. $\square$

Proof of Theorem 1. Just as in the proof of Lemma 3, use Theorem 2 to show that $\text{Alg } \mathcal{P}$ embeds in $(\sum \oplus \mathcal{A}_n)_{\infty}$ completely complemented for some finite-dimensional nest algebras $\mathcal{A}_n$'s.

Let $\widetilde{\mathcal{P}}$ be a continuous nest of multiplicity infinity. We clearly have that $(\sum \oplus \mathcal{A}_n)_{\infty}$ embeds in $\text{Alg } \widetilde{\mathcal{P}}$ completely complemented. The similarity Theorem (see [L]) tells us that $\text{Alg } \widetilde{\mathcal{P}}$ is similar to a continuous nest of multiplicity one, and the latter is completely isomorphic to $(\sum \oplus \mathcal{T}_n)_{\infty}$. Therefore, $\text{Alg } \mathcal{P}$ embeds completely bounded and completely complemented in $(\sum \oplus \mathcal{T}_n)_{\infty}$. $\square$

Remark. $(\sum \oplus \mathcal{T}_n)_{\infty}$ does not have the approximation property. Notice that M_n 1-embeds into $\mathcal{T}_{2n}$ (in the upper right corner); therefore, $(\sum \oplus M_n)_{\infty}$ 1-embeds into $(\sum \oplus \mathcal{T}_n)_{\infty}$. Since $B(H)$ is isomorphic to $(\sum \oplus M_n)_{\infty}$ and does not have the approximation property (see [S]), the result follows.

Remark. $(\sum \oplus \mathcal{T}_n)_{\infty}$ is not isomorphic to $B(H)$. The result is analogous to the one that says that H^{∞} is not isomorphic to L_{∞} , (see [Pe2]) but is deeper. We are indebted to G. Pisier for pointing it out to us. It follows from a result of his that says that H^{∞} is not isomorphic to a quotient of a C^* -algebra (see [DGT]); whereas H^{∞} is isomorphic to a quotient of $\text{Alg } \mathcal{P}$, the discrete nest algebra (see [Po]).

The weaker statement $(\sum \oplus \mathcal{T}_n)_{\infty}$ is not completely isomorphic to $B(H)$, is easy because $(\sum \oplus \mathcal{T}_n)_{\infty}$ is not a λ -injective operator space for any $\lambda \geq 1$. If it were, then the main triangular projection on M_n would be uniformly bounded.

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